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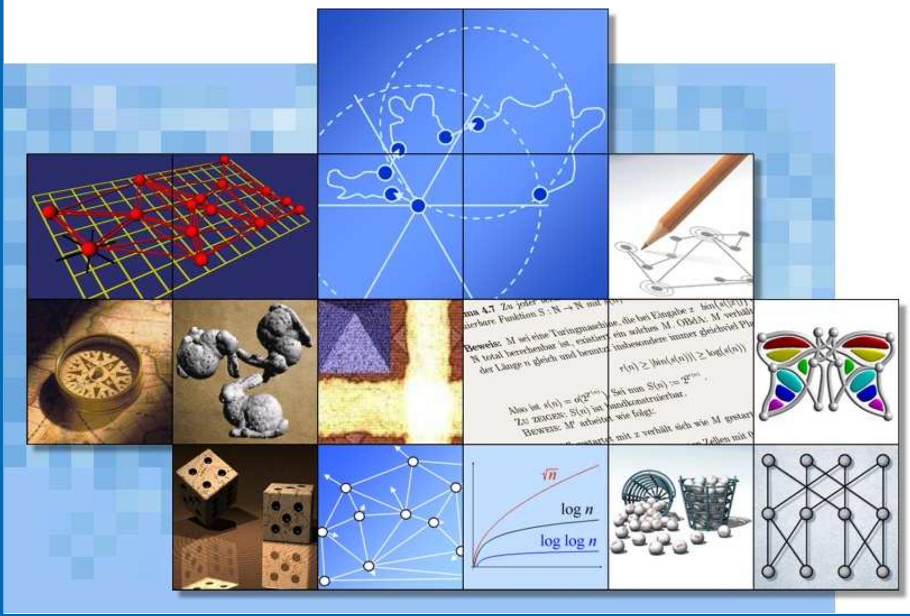
University of Paderborn
Algorithms and Complexity

Average case complexity of Voronoi diagrams of n sites from the unit cube



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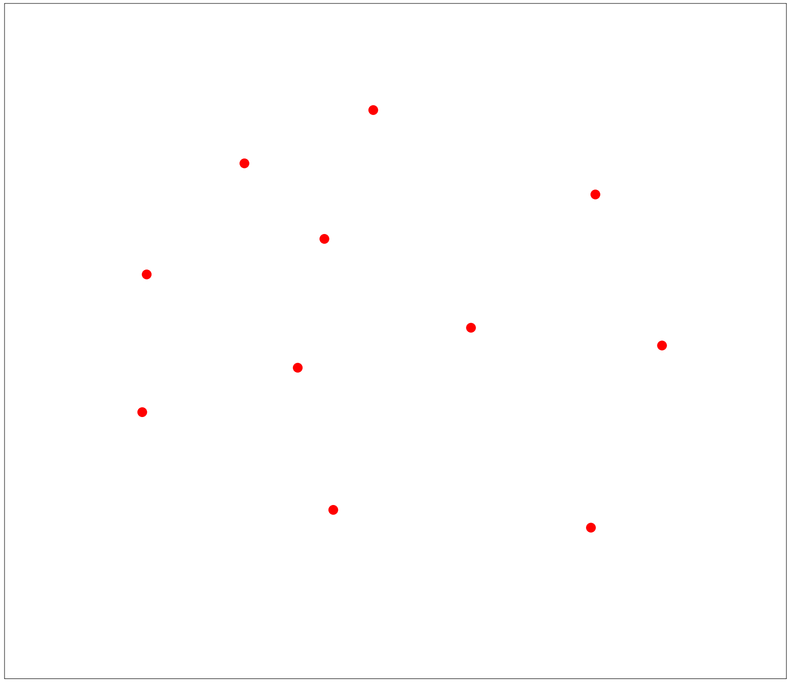
EWCG'05

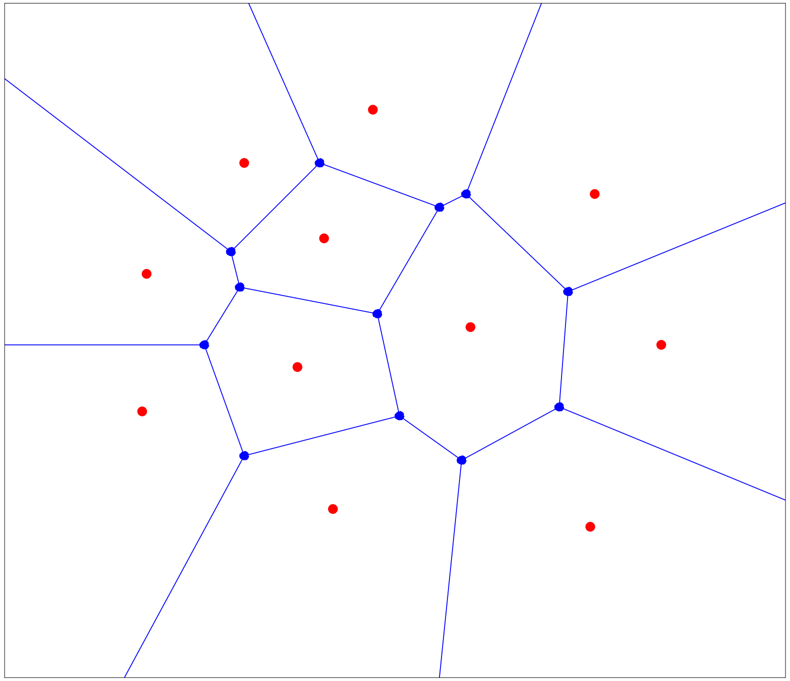
Introduction

- Given a set of point sites S in $\mathbb{R}^d$
  (in general position, i.e., no $d+2$ sites lie on a common d-sphere)



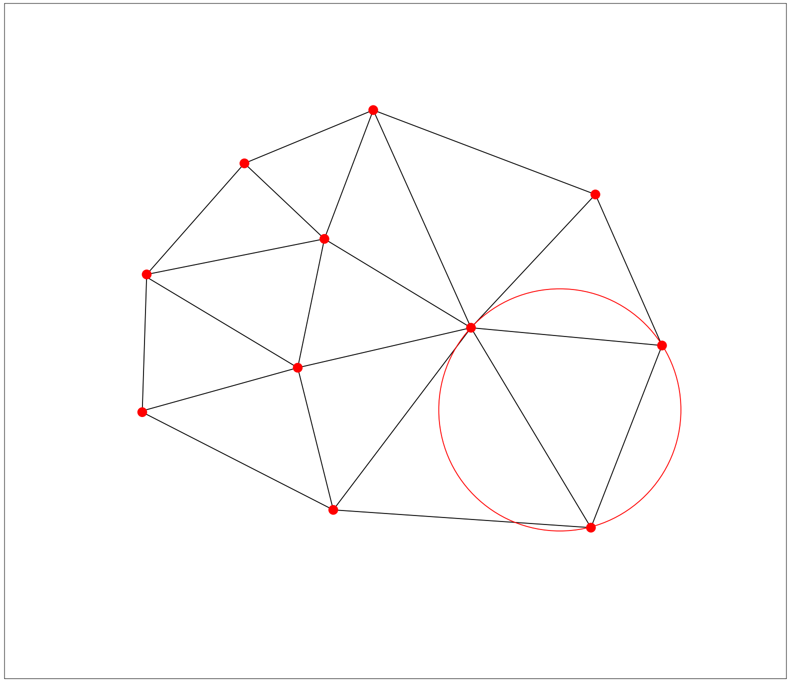
Voronoi diagram

- Partition of $\mathbb{R}^d$ into d-polyhedra, one per site, called Voronoi cells
- Voronoi cell: all points of $\mathbb{R}^d$ strictly closer to the site $s \in S$ than to any other site in S



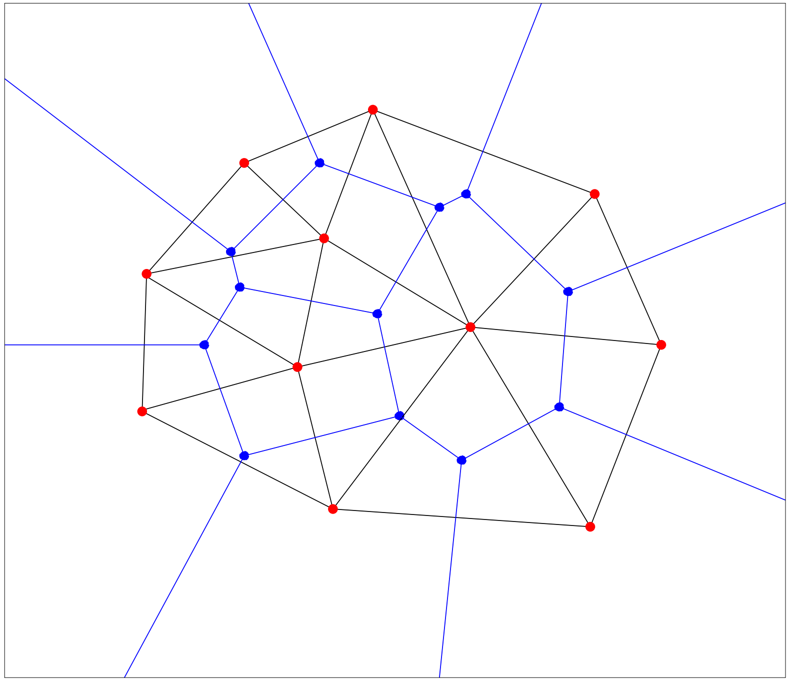
Delaunay triangulation

- Partition of the convex hull of S into d-simplices, called Delaunay cells, with vertices from S $\longrightarrow$ vertices = sites
- Circumball of each d-simplex has no other sites in its interior



Duality

Delaunay triangle	⟷	Voronoi vertex
Delaunay edge	⟷	Voronoi edge
Delaunay vertex (site)	⟷	Voronoi cell



Complexity

Combinatorial complexity of $\left\{ \begin{array}{c} \text{Voronoi diagram} \\ \text{Delaunay triangulation} \end{array} \right\}$

= number of all $\left\{ \begin{array}{c} \text{Voronoi} \\ \text{Delaunay} \end{array} \right\}$ faces

$$= \left\{ \begin{array}{l} O(\text{number of Voronoi vertices}) \\ O(\text{number of Delaunay simplices}) \end{array} \right.$$

where the dimension d is considered to be a constant!

Worst Case and Average Case

- **Worst case complexity**

$$\mathcal{O}\left(n^{\lceil d/2 \rceil}\right)$$

[Klee, 1980] [Seidel, 1987]

- **Average case complexity**
  - For n i.i.d. random point sites chosen uniformly from a d-ball

$$\mathcal{O}(n)$$

  - Conjecture: similar bounds hold for any uniform distribution in a convex domain

[Dwyer, 1989 & 1991]

Our result

Theorem

For n i.i.d. random point sites chosen uniformly from the unit d-hypercube $[0,1]^d$

$$\mathbf{E}\left[\text{number of Delaunay simplices}\right] = O(n)$$

where the dimension d is considered to be a constant.

- Outline
  - some preliminary ideas
  - a crucial lemma (without proof)
  - proof sketch for the theorem
- From now on: all random points are meant to be i.i.d. random points chosen uniformly from the unit d-hypercube

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Average Analysis

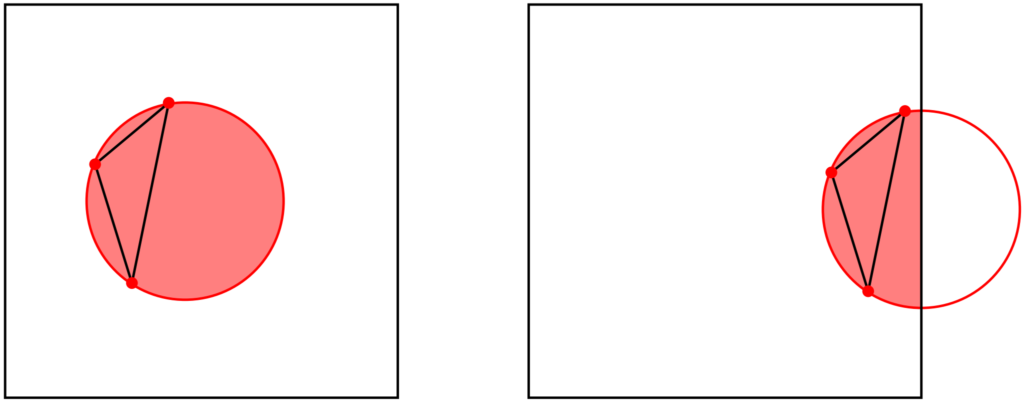
- Generally,

$$\mathbf{E}\left[\text{number of Delaunay simpl.}\right] = \binom{n}{d+1} \cdot \mathbf{Pr}\left[\text{circumball}(\Delta) \text{ is empty}\right]$$

where Δ is a “random” d-simplex and circumball(Δ) is the smallest d-ball enclosing Δ

- Unfortunately,

$$\mathbf{Pr}\left[\text{circumball}(\Delta) \text{ is empty}\right] \neq \left(1 - \text{vol}(\text{circumball}(\Delta))\right)^{n-(d+1)}$$



Crucial Lemma

- We need to bound

$$\text{vol}\left(\text{circumball}(\Delta) \cap [0,1]^d\right)$$

Lemma

Let Δ be a random d-simplex in $[0,1]^d$ (i.e., it is the convex hull of $d+1$ i.i.d. random point sites chosen uniformly from $[0,1]^d$)

$$\mathbf{Pr}\left[\text{vol}\left(\text{circumball}(\Delta) \cap [0,1]^d\right) \leq a\right] \leq \text{const}_d \cdot a^d$$

where const_d is a constant depending only on d.

Proof of the theorem (1)

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- There are $\binom{n}{d+1}$ possible simplices
- Consider (classes of) simplices with “large” circumball
  - it’s likely that they have another point site in their circumball
    $\longrightarrow$ only few Delaunay simplices
- Remaining simplices with “small” circumball are also very few
- $\Rightarrow$ only few Delaunay simplices at all

Proof of the theorem (2)

- Classes of simplices $S_0, \ldots, S_{\log n-1}$ with large circumball

$$\Delta \in S_i \quad \Leftrightarrow \quad \frac{1}{2^{i+1}} < \text{vol}\left(\text{circumball}(\Delta) \cap [0,1]^d\right) \leq \frac{1}{2^i}$$

- With the crucial lemma

$$\mathbf{Pr}\left[\Delta \in S_i\right] \leq \text{const}_d \cdot \left(\frac{1}{2^i}\right)^d$$

- For $\Delta \in S_i$ it is

$$\mathbf{Pr}\left[\text{circumball}(\Delta) \text{ is empty} \mid \Delta \in S_i\right] \leq \left(\frac{1}{2}\right)^{\frac{n-(d+1)}{2^{i+1}}}$$

Proof of the theorem (3)

- $\mathbf{E}\left[\text{number of Delaunay simplices} \in S_i\right]$

$$\leq \binom{n}{d+1} \cdot \mathbf{Pr}\left[\Delta \in S_i\right] \cdot \mathbf{Pr}\left[\text{circumball}(\Delta) \text{ is empty } \middle| \Delta \in S_i\right]$$

$$\leq \binom{n}{d+1} \cdot \text{const}_d \cdot \left(\frac{1}{2^i}\right)^d \cdot \left(\frac{1}{2}\right)^{\frac{n-(d+1)}{2^{i+1}}}$$

- Expected number of Delaunay simplices with large circumball

$$\sum_{i=0}^{\log n - 1} \mathbf{E}\left[\text{number of Delaunay simplices} \in S_i\right] = \mathcal{O}(n)$$

Proof of the theorem (4)

- Class of remaining simplices with small circumball

$$\Delta \in S_{re} \quad \Leftrightarrow \quad \mathrm{vol}\left(\mathrm{circumball}(\Delta) \cap [0,1]^d\right) \le \frac{1}{n}$$

- With the crucial lemma

$$\mathbf{E}\left[\text{number of simplices } \in S_{re}\right] \le \mathrm{const}_d \cdot n$$

- $\mathbf{E}\left[\text{number of Delaunay cells}\right]$

$$\le \sum_{i=0}^{\log n - 1} \mathbf{E}\left[\text{number of Delaunay simplices} \in S_i\right]$$

$$+\, \mathbf{E}\left[\text{number of simplices } \in S_{re}\right] \quad = \quad O(n)$$

Conclusion

- Average case complexity of Voronoi diagrams for the unit hypercube is linear
- Problem open since more than 15 years
- Future work: “smoothed” complexity of Voronoi diagrams

Thanks!