Average case complexity of Voronoi diagrams of n sites from the unit cube[*]

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Abstract

We consider the expected number of Voronoi vertices (or number of Delaunay cells for the dual structure) for a set of n i.i.d. random point sites chosen uniformly from the unit d-hypercube $[0,1]^d$. We show an upper bound for this number which is linear in n, the number of random point sites, where d is assumed to be a constant. This result matches the trivial lower bound of n.

This is an open problem since several years. In 1991, Dwyer [2] showed that for a uniform distribution from the unit d-ball the average number of Voronoi vertices is linear in n and it is commonly assumed that this holds for any reasonable probability distribution.

1 Introduction

Voronoi diagrams are a fundamental structure in several fields of science besides mathematics and computer science such as physics, geology, agriculture, geography, etc. Named after the Russian mathematician Voronoi [10] they have been 'reinvented' by other researchers, e.g., by the physicists Wigner and Seits [11] or the meteorologist Thiessen [9].

The Voronoi diagram of a set $\mathcal{S}$ of n points – called sites – partitions space into n regions, one per site. The region of a site s consists of all points that are closer to s than to any other site. The straight-line dual of the Voronoi diagram in the plane and its extension to higher dimensions is called the *Delaunay triangulation*. A triangulation of a set $\mathcal{S}$ of sites is a complete partition of the convex hull of $\mathcal{S}$ into fully dimensional simplices having the sites as vertices. The Delaunay triangulation is the unique triangulation of the set of sites such that the circumsphere of every simplex contains no other site in its interior. The Voronoi diagram can be computed in linear time from the Delaunay triangulation, using the one-to-one correspondence between their faces.

Voronoi diagrams have the great advantage to be a rather simple but quite elegant structure with many extensions obtained by varying metric, sites, environment, and constraints. In computer science they are widely used in clustering, mesh generation, graphics, curve and surface reconstruction, and other applications.

A vast variety of basic and (relatively) simple algorithms exists for their construction such as the plane sweep, the divide-and-conquer, the incremental, and the gift-wrapping algorithm, see also Chapter 20 in the Handbook of Discrete and Computational Geometry [5]. In fact most of these algorithms are actually specialized convex hull algorithms since there is a close connection with convexity. Any $(d+1)$-dimensional convex hull algorithm can be used to compute a d-dimensional Delaunay triangulation. All these algorithms depend in their run time on the number of faces of the Delaunay triangulation. Unfortunately, in d dimensions this number is $\Theta(n^{\lceil d/2 \rceil})$ in the worst case [7, 8] (for the usual diagram with the Euclidean metric).

Recent research attempts to quantify situations when the complexity of the Voronoi diagram is low or when it is high [4]. The average case complexity was considered by Dwyer [2] who showed that for n i.i.d. random point sites chosen uniformly from the unit d-ball the expected number of Delaunay simplices is $\Theta(n)$. It has been conjectured that this bound also holds for any uniform distribution in a convex domain but until now no explicit proofs were given [2, 6].

For further reading on Voronoi diagrams and Delaunay triangulations we refer to the survey by Franz Aurenhammer [1], the book by Herbert Edelsbrunner [3] and Chapter 20 in the Handbook of Discrete and Computational Geometry [5].

2 Average case

In this section we will present an average case analysis for the number of Delaunay cells. Let $\mathcal{P}$ be a set of n i.i.d. random points chosen uniformly from the unit d-hypercube $[0,1]^d$. Let $\mathbf{D}(\mathcal{P})$ be the Delaunay triangulation of $\mathcal{P}$. Generally, we will use that

$$\mathbf{E}\big[\text{number of Delaunay simplices of } \mathbf{D}(\mathcal{P})\big] = \binom{n}{d+1} \cdot \mathbf{Pr}\big[\,\text{c-ball}(\Delta) \text{ is empty}\big]$$

where Δ is a random d-simplex, i.e., it is the convex hull of $d+1$ random point sites chosen uniformly from $[0,1]^d$,

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and $\text{c-ball}(\Delta)$ is the smallest d-ball enclosing Δ.

Unfortunately, in general it is

$$\mathbf{Pr}\big[\,\text{c-ball}(\Delta)\text{ is empty}\big] \neq \big(1-\text{vol}(\text{c-ball}(\Delta))\big)^{n-(d+1)}$$

for the following reason. All random point sites are from inside $[0,1]^d$ while some part of $\text{c-ball}(\Delta)$ might lie outside of $[0,1]^d$. Of course, the probability for a random point site to be in the outer part of $\text{c-ball}(\Delta)$ is equal to 0 and therefore we must not consider the outer part. This causes the main difficulty in our analysis namely to bound the volume of $\text{c-ball}(\Delta)\cap[0,1]^d$.

Fortunately, we can show the following crucial lemma though we postpone the proof to Section 3.

Lemma 1 *Let Δ be a random d-simplex, i.e., its $d+1$ vertices are i.i.d. random points chosen uniformly from $[0,1]^d$. Then for any constant $a\in[0,1]$ it is*

$$\mathbf{Pr}\left[\text{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\leq a\right] \;\leq\; \text{const}_d\cdot a^d$$

where const_d is a constant depending only on d.

Based on this lemma we will now establish the main theorem of this section.

Theorem 2 *For n i.i.d. random points sites chosen uniformly from $[0,1]^d$ it holds that*

$$\mathbf{E}\big[\textit{number of Delaunay simplices}\big] \;=\; \mathcal{O}(n)\ .$$

Proof. The main idea of the proof is to consider (classes of) simplices with a 'large' circumball that are very likely to have another point site in their circumball, i.e., these simplices are not Delaunay simplices. Then we show that the remaining simplices with a 'small' circumball are very few.

Let us assume w.l.o.g. that n is a power of 2. Let us consider the $\binom{n}{d+1}$ possible simplices that have $d+1$ of the given n random point sites as vertices. For the simplices with 'large' circumball we define classes $\mathcal{S}_0,\ldots,\mathcal{S}_{\log n-1}$ s.t. for a simplex Δ we have that

$$\Delta\in\mathcal{S}_i \;\Leftrightarrow\; \frac{1}{2^{i+1}} \;<\; \text{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right) \;\leq\; \frac{1}{2^i}\ .$$

From Lemma 1 it follows immediately that

$$\begin{aligned}\mathbf{Pr}\big[\Delta\in\mathcal{S}_i\big] \;&\leq\; \mathbf{Pr}\left[\text{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\leq\frac{1}{2^i}\right]\\ &\leq\; \text{const}_d\cdot\left(\frac{1}{2^i}\right)^d\ .\end{aligned}$$

The probability for a simplex $\Delta\in\mathcal{S}_i$ to be a Delaunay simplex is

$$\begin{aligned}\mathbf{Pr}\big[\,\text{c-ball}(\Delta)\text{ is empty}\mid\Delta\in\mathcal{S}_i\big] \;&\leq\; \left(1-\frac{1}{2^{i+1}}\right)^{n-(d+1)}\\ \leq\; \left(\frac{1}{e}\right)^{\frac{n-(d+1)}{2^{i+1}}} \;&\leq\; \left(\frac{1}{2}\right)^{\frac{n-(d+1)}{2^{i+1}}}\ .\end{aligned}$$

Now we can bound the expected number of Delaunay simplices for each class $\mathcal{S}_i$.

For $0\leq i\leq\log n-1$ it is

$$\begin{aligned}&\mathbf{E}\big[\text{number of Delaunay simplices}\in\mathcal{S}_i\big]\\ &\quad\leq\; \binom{n}{d+1}\cdot\mathbf{Pr}\big[\Delta\in\mathcal{S}_i\big]\\ &\qquad\qquad\cdot\mathbf{Pr}\big[\,\text{c-ball}(\Delta)\text{ is empty}\mid\Delta\in\mathcal{S}_i\big]\\ &\quad\leq\; \binom{n}{d+1}\cdot\text{const}_d\cdot\left(\frac{1}{2^i}\right)^d\cdot\left(\frac{1}{2}\right)^{\frac{n-(d+1)}{2^{i+1}}}\ .\end{aligned}$$

The expected number of Delaunay simplices for all classes $\mathcal{S}_0,\ldots,\mathcal{S}_{\log n-1}$ is

$$\begin{aligned}&\sum_{i=0}^{\log n-1}\mathbf{E}\big[\text{number of Delaunay simplices}\in\mathcal{S}_i\big]\\ &\leq\; \binom{n}{d+1}\cdot\text{const}_d\sum_{i=0}^{\log n-1}\left(\frac{1}{2}\right)^{i\cdot d+\frac{n-(d+1)}{2^{i+1}}}\\ &=\; \binom{n}{d+1}\cdot\text{const}_d\sum_{i=0}^{\log n-1}\left(\frac{1}{2}\right)^{(\log n-(i+1))\cdot d+\frac{n-(d+1)}{2^{\log n-(i+1)+1}}}\\ &=\; \binom{n}{d+1}\cdot\text{const}_d\cdot\frac{1}{n^d}\sum_{i=0}^{\log n-1}\left(\frac{1}{2}\right)^{2^i\cdot\left(1-\frac{d+1}{n}\right)-(i+1)\cdot d}\\ &\leq\; n\cdot\text{const}_d\cdot\Big((d+2)\cdot2^{(d+3)\cdot d}+1\Big) \;=\; \mathcal{O}(n)\ .\end{aligned}$$

The last step follows immediately if $d+2\geq\log n-1$ since

$$\begin{aligned}\sum_{i=0}^{d+2}\left(\frac{1}{2}\right)^{2^i\cdot\left(1-\frac{d+1}{n}\right)-(i+1)\cdot d} \;&\leq\; \sum_{i=0}^{d+2}2^{(i+1)\cdot d}\\ &\leq\; (d+2)\cdot2^{(d+3)\cdot d}\ .\end{aligned}$$

In the other case it is

$$\sum_{i=d+3}^{\log n-1}\left(\frac{1}{2}\right)^{2^i\cdot\left(1-\frac{d+1}{n}\right)-(i+1)\cdot d} \;\leq\; \sum_{i=d+3}^{\log n-1}\left(\frac{1}{2}\right)^i \;\leq\; 1\ ,$$

where we assume that $n\geq2\cdot(d+1)$.

The expected number of remaining simplices with 'small' circumball can be bounded using Lemma 1, too. Let $\mathcal{S}_{\text{re}}$ denote the set of simplices s.t. for a simplex Δ we have

$$\Delta\in\mathcal{S}_{\text{re}} \;\Leftrightarrow\; \text{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right) \;\leq\; \frac{1}{n}\ .$$

Then it is

$$\begin{aligned}&\mathbf{E}\big[\text{number of simplices }\in\mathcal{S}_{\text{re}}\big]\\ &\quad\leq\; \binom{n}{d+1}\cdot\mathbf{Pr}\left[\text{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\leq\frac{1}{n}\right]\\ &\quad\leq\; n^{d+1}\cdot\text{const}_d\cdot\frac{1}{n^d} \quad\leq\quad n\cdot\text{const}_d\ .\end{aligned}$$

Now we can combine everything and by linearity of expectation we get that

$$
\begin{aligned}
&\mathbf{E}\left[\text{number of Delaunay cells}\right] \\
&\quad\leq \sum_{i=0}^{\log n-1} \mathbf{E}\left[\text{number of Delaunay simplices} \in \mathcal{S}_i\right] \\
&\qquad\qquad + \mathbf{E}\left[\text{number of simplices } \in \mathcal{S}_{\mathrm{re}}\right] \\
&\quad\leq n \cdot \mathrm{const}_d \cdot \left((d+2)^{(d+3)\cdot d} + 2\right) \quad = \quad \mathcal{O}(n) \ ,
\end{aligned}
$$

which concludes the proof of Theorem 2. ☐

3 Proof of Lemma 1

Let $p_1, \ldots, p_{d+1} \in [0,1]^d$ be the vertices of simplex $\Delta = \Delta(p_1, \ldots, p_{d+1})$, i.e., Δ is the convex hull of $p_1, \ldots, p_{d+1}$. The volume of $\text{c-ball}(\Delta)$ is given by $\mathcal{V}_d \cdot r^d$ where $r = r(\Delta)$ is the radius of the circumball of Δ and $\mathcal{V}_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of the unit d-ball. We can approximate the radius $r(\Delta)$ and the volume of $\text{c-ball}(\Delta)$ by the following observation:

Observation 1 *It holds that*

$$
\begin{aligned}
2 \cdot r(\Delta) \quad &\geq \quad \max_{1 \leq i < j \leq d+1} \|p_i - p_j\|_2 \\
&\geq \quad \max_{1 \leq i < j \leq d+1} \|p_i - p_j\|_\infty \\
&=: \quad \mathrm{maxwidth}(\Delta)
\end{aligned}
$$

and therefore it is

$$
\mathrm{vol}\left(\text{c-ball}(\Delta)\right) \ \geq \ \mathcal{V}_d \cdot \frac{1}{2^d} \cdot \mathrm{maxwidth}(\Delta)^d \ .
$$

In other words we approximate the volume of $\text{c-ball}(\Delta)$ by a fraction of the volume of a smallest hypercube containing all the point sites $p_1, \ldots, p_{d+1}$, cf. also Figure 1.

In a next step we will reformulate our random process. Instead of considering $d+1$ many d-dimensional random variables (= point sites) we will combine the random variables coordinate-wise leading to d sets of $d+1$ random numbers each. In more detail, let us again consider the point sites $p_1, \ldots, p_{d+1} \in [0,1]^d$ where $p_i = (p_i^{(1)}, \ldots, p_i^{(d)})$ for $1 \leq i \leq d+1$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_d$ be the sets s.t. $\mathcal{P}_j = \{p_1^{(j)}, \ldots, p_{d+1}^{(j)}\}$ for $1 \leq j \leq d$ and let

$$
\mathrm{width}(\mathcal{P}_j) \ := \ \max \mathcal{P}_j - \min \mathcal{P}_j
$$

denote the maximal distance between two elements in $\mathcal{P}_j$. We can now define the variable $\mathrm{maxwidth}$ in another way as

$$
\mathrm{maxwidth}(\mathcal{P}_1, \ldots, \mathcal{P}_d) \ := \ \max_{1 \leq j \leq d} \mathrm{width}(\mathcal{P}_j) \ ,
$$

which is consistent with the earlier definition, i.e., it is $\mathrm{maxwidth}(\Delta) = \mathrm{maxwidth}(\mathcal{P}_1, \ldots, \mathcal{P}_d)$. (Therefore we sometimes write only $\mathrm{maxwidth}$.)

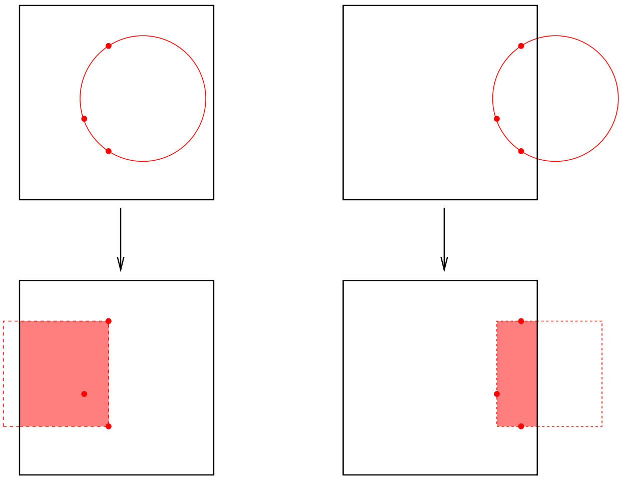

Figure 1: The 2 dimensional case: 3 point sites and their circumcircles in the unit square. Generally, the position of the smallest hypercube containing all point sites is not uniquely defined. When intersected by $[0,1]^d$ we consider the hypercube that has smallest intersection volume.

Since we actually want to bound the volume of $\text{c-ball}(\Delta) \cap [0,1]^d$, we consider the (smallest) hypercube containing all point sites that has minimal volume when intersected by $[0,1]^d$. Therefore, we introduce the variable value that indicates how much each dimension contributes to the volume of the minimal intersection between a smallest hypercube containing all the point sites and $[0,1]^d$. If for a fixed dimension the coordinates of all point sites lie close to 0 (or 1) then the dimension contributes less than $\mathrm{maxwidth}$ to the volume, namely only the distance of the maximal coordinate to 0 (or the minimal coordinate to 1), cf. also Figure 1. In other words, the hypercube then sticks out of $[0,1]^d$ in this dimension.

We define now the value of set $\mathcal{P}_j$ to be

$$
\mathrm{value}(\mathcal{P}_j) \ := \ \begin{cases} \mathrm{maxwidth}(\mathcal{P}_1, \ldots, \mathcal{P}_d) & \\ \qquad \textbf{if } \max \mathcal{P}_j - \mathrm{maxwidth} \geq 0 \\ \qquad \text{and } \min \mathcal{P}_j + \mathrm{maxwidth} \leq 1 \\ \min\{\max \mathcal{P}_j, 1 - \min \mathcal{P}_j\} \quad \textbf{else} \ . \end{cases}
$$

With these definitions we can formulate the following lemma.

Lemma 3 *It holds that*

$$
\begin{aligned}
&\mathrm{vol}\left(\text{c-ball}(\Delta) \cap [0,1]^d\right) \\
&\quad \geq \ \min\left\{\mathcal{V}_d \cdot \left(\frac{1}{2}\right)^{2d}, \frac{1}{d!}\right\} \cdot \prod_{j=1}^{d} \mathrm{value}(\mathcal{P}_j) \ .
\end{aligned}
$$

Due to space limitations we defer the proof of Lemma 3 to a later full version of this paper.

From now on we will consider the following random process: we have d sets $\mathcal{P}_1, \ldots, \mathcal{P}_d$ of $d+1$ i.i.d. random numbers chosen uniformly from the interval $[0,1]$. From Lemma 3 it follows that if we show that

$$\mathbf{Pr}\left[\prod_{j=1}^{d} \mathrm{value}(\mathcal{P}_j) \le a\right] \;=\; \mathcal{O}\left(a^d\right) \quad , \tag{1}$$

then Lemma 1 is also shown.

In order to show (1) we will now establish two lemmas. The first one covers the case that $\mathrm{maxwidth}$ is smaller than $\sqrt[d]{a}$, then it follows immediately that $\prod_{j=1}^{d} \mathrm{value}(\mathcal{P}_j) \le a$. The second lemma covers the case that $\mathrm{maxwidth}$ is larger than $\sqrt[d]{a}$ and we have to spend some more effort to show (1).

Lemma 4 *For any value* $a \in [0,1]$ *it holds that*

$$\mathbf{Pr}\left[\mathrm{maxwidth}(\mathcal{P}_1,\dots,\mathcal{P}_d) \le \sqrt[d]{a}\right] \;=\; \mathcal{O}\left(a^d\right) \quad .$$

Proof. It suffices to bound the probability that $\mathrm{width}(\mathcal{P}_j) \le \sqrt[d]{a}$ for $1 \le j \le d$. For set $\mathcal{P}_j$ we fix the two elements with the maximal distance, i.e., we fix $\max \mathcal{P}_j$ and $\min \mathcal{P}_j$ where the distance between both must not exceed $\sqrt[d]{a}$. The remaining $d-1$ elements in $\mathcal{P}_j$ must have values between $\max \mathcal{P}_j$ and $\min \mathcal{P}_j$. Now we can write

$$\mathbf{Pr}\left[\mathrm{width}(\mathcal{P}_j) \le \sqrt[d]{a}\right] \;\le\; (d+1)\cdot d \cdot \int_0^1 \int_{\max\{0, Y-\sqrt[d]{a}\}}^{Y} (Y-X)^{d-1} \;\mathrm{d}X\;\mathrm{d}Y \tag{2}$$

where the outer intergral denotes the range of element $\max \mathcal{P}_j (= Y)$ and the inner integral the range of element $\min \mathcal{P}_j (= X)$. The integration boundaries assure that their distance is at most $\sqrt[d]{a}$. The integrand $(Y-X)^{d-1}$ denotes exactly the probability that all remaining $d-1$ elements of $\mathcal{P}_j$ are between Y and X. The factor before the integral is due to fixing the maximal and minimal element in $\mathcal{P}_j$.

In order to solve this integral we will split it up in the following way to remove the maximum expression from the integration boundary of the inner integral.

$$\begin{aligned}
&\int_0^1 \int_{\max\{0, Y-\sqrt[d]{a}\}}^{Y} (Y-X)^{d-1} \;\mathrm{d}X\;\mathrm{d}Y \\
=\;& \int_0^{\sqrt[d]{a}} \int_0^{Y} (Y-X)^{d-1} \;\mathrm{d}X\;\mathrm{d}Y \\
&+ \int_{\sqrt[d]{a}}^{1} \int_{Y-\sqrt[d]{a}}^{Y} (Y-X)^{d-1} \;\mathrm{d}X\;\mathrm{d}Y \\
=\;& \frac{1}{d}\cdot\left(\int_0^{\sqrt[d]{a}} Y^d \;\mathrm{d}Y \;+\; \int_{\sqrt[d]{a}}^{1} a \;\mathrm{d}Y\right) \\
=\;& \frac{1}{d}\cdot\left(\frac{1}{d+1}\cdot a^{\frac{d+1}{d}} + a - a^{\frac{d+1}{d}}\right) \quad\le\quad \frac{1}{d}\cdot a
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbf{Pr}\left[\mathrm{width}(\mathcal{P}_j) \le \sqrt[d]{a}\right] \;\le\; (d+1)\cdot a \quad\Rightarrow \\
&\qquad \mathbf{Pr}\left[\mathrm{maxwidth} \le \sqrt[d]{a}\right] \;=\; \mathcal{O}\left(a^d\right) \quad .
\end{aligned}$$

□

Lemma 5 *For any value of* $a \in [0,1]$ *it holds that*

$$\mathbf{Pr}\left[\mathrm{maxwidth}(\mathcal{P}_1,\dots,\mathcal{P}_d) > \sqrt[d]{a} \quad \textit{and} \quad \prod_{j=1}^{d} \mathrm{value}(\mathcal{P}_j) \le a\right] \;\le\; \mathcal{O}\left(a^d\right) \quad .$$

The proof of Lemma 5 is very involved and rather lengthy. We defer it also to a full version of this paper.

From Lemma 4 and Lemma 5 it follows that Equation (1) holds and thus Lemma 1 is shown.

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Appendix

Proof of Lemma 3

Lemma 3 *It holds that*

$$\operatorname{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\ \geq\ \min\left\{\mathcal{V}_d\cdot\left(\frac{1}{2}\right)^{2d},\frac{1}{d!}\right\}\cdot\prod_{j=1}^{d}\text{value}(\mathcal{P}_j)\ .$$

Proof. We will consider two cases, namely that the center of c-ball(Δ) lies inside $[0,1]^d$ (case i) and the case that it doesn't (case ii).

(i) If the center of c-ball(Δ) lies inside $[0,1]^d$ it follows immediately that at least a fraction of $(1/2)^d$ of c-ball(Δ) lies inside $[0,1]^d$. Together with note 1 it is

$$\begin{aligned}\operatorname{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right) &\geq \left(\frac{1}{2}\right)^d\cdot\operatorname{vol}\left(\text{c-ball}(\Delta)\right)\\ &\geq \left(\frac{1}{2}\right)^d\cdot\mathcal{V}_d\cdot\left(\frac{\text{maxwidth}}{2}\right)^d\\ &\geq \left(\frac{1}{2}\right)^{2d}\cdot\mathcal{V}_d\cdot\prod_{j=1}^{d}\text{value}(\mathcal{P}_j)\ .\end{aligned}$$

(ii) Let $\mathcal{B}=\mathcal{B}(\text{c-ball}(\Delta)\cap[0,1]^d)$ be the smallest axis parallel box containing c-ball$(\Delta)\cap[0,1]^d$ and let $b_1,\ldots,b_d$ be the width of box $\mathcal{B}$ in each dimension, i.e., $\operatorname{vol}(\mathcal{B})=\prod_{i=1}^d b_i$. In a first step we want to show that

$$d!\cdot\operatorname{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\ \geq\ \operatorname{vol}(\mathcal{B})\ . \tag{3}$$

We know that

$$\operatorname{vol}(\mathcal{B})=\left|\det\begin{pmatrix}\bar{0} & b_1\cdot e_1 & \ldots & b_d\cdot e_d\\ 1 & 1 & \ldots & 1\end{pmatrix}\right|$$

where $\bar{0}=(0,\ldots,0)^T$ is the origin and $e_1,\ldots,e_d$ are unit vectors.

W.l.o.g. we assume that $\bar{0}$ is a vertex of $\mathcal{B}$.
Consider now the case that c-ball$(\Delta)\cap[0,1]^d$ contains at least $d+1$ vertices of $\mathcal{B}$ and let these vertices be $\bar{0},b_1\cdot e_1,\ldots,b_d\cdot e_d$. The convex hull of these vertices, namely the simplex $\mathcal{S}:=\operatorname{conv}(\bar{0},b_1\cdot e_1,\ldots,b_d\cdot e_d)$ is then completely containd in c-ball$(\Delta)\cap[0,1]^d$ and therefore it is $\operatorname{vol}\left(\text{c-ball}(\Delta)\cap[0,1]^d\right)\geq\operatorname{vol}(\mathcal{S})$. From geometry we knwo that

$$\operatorname{vol}(\mathcal{S})=\frac{1}{d!}\cdot\operatorname{vol}(\mathcal{B})$$

and therefore (3) follows.
Let us now consider the case that c-ball$(\Delta)\cap[0,1]^d$ does not contain $d+1$ vertices of $\mathcal{B}$. Our goal is to place a collection of simplices inside c-ball$(\Delta)\cap[0,1]^d$ such that their sum of volumes is also equal to $1/d!\cdot\operatorname{vol}(\mathcal{B})$.
Let $\mathcal{C}:=(c_1,\ldots,c_d)$ be the center of c-ball(Δ) and consider then the d hyperplanes $X_1=c_1,\ldots,X_d=c_d$ that subdivide c-ball(Δ) into its 2^d equal parts. These hyperplanes subdivide also $\mathcal{B}$ into at most 2^{d-1} smaller boxes (remember, the center $\mathcal{C}$ is not contained in $[0,1]^d$). For each of these smaller boxes we will find at least $d+1$ vertices that lie also in c-ball$(\Delta)\cap[0,1]^d$. As before we can take their convex hull to construct the desired simplices, it follows also immediately that their sum of volumes is equal to $1/d!\cdot\operatorname{vol}(\mathcal{B})$. Thus (3) is shown, cf. also figure.

In the next step we want to show that

$$\operatorname{vol}(\mathcal{B})\ =\ \prod_{i=1}^{d}b_i\ \geq\ \prod_{j=1}^{d}\text{value}(\mathcal{P}_j)\ . \tag{4}$$

Again we will consider two cases, namely that c-ball$(\Delta)\cap[0,1]^d$ and therefore $\mathcal{B}$ contains at least one vertex of $[0,1]^d$ (case a) and that it doesn't (case b). We will show now the following claim

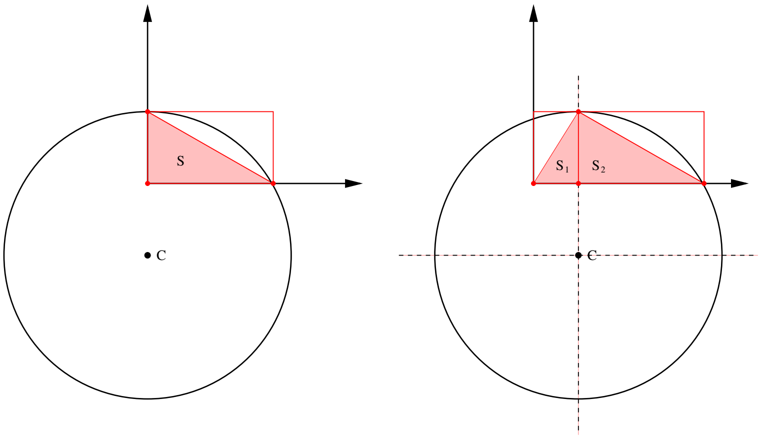

Figure 2: In two dimensions: if three of the vertices of $\mathcal{B}$ belong also to $\text{c-ball}(\Delta)$ the construction of $\mathcal{S}$ is straight forward. In the other case two simplices $\mathcal{S}_1$ and $\mathcal{S}_2$ are constructed in the depicted way.

Claim: It holds that $\quad \text{value}(\mathcal{P}_j) \leq b_j \quad$ *for all* $1 \leq j \leq d$.

Proof of Claim:

(a) Let us assume that $\text{c-ball}(\Delta) \cap [0,1]^d$ contains a vertex of $[0,1]^d$, say $(v_1, \ldots, v_d)$ where $v_j \in \{0,1\}$ for $1 \leq j \leq d$. Since $\mathcal{B}$ contains all vertices of the simplex Δ we can conclude that either $\max \mathcal{P}_j \;\leq\; b_j$ if $v_j = 0$ or that $1 - \min \mathcal{P}_j \;\leq\; b_j$ if $v_j = 1$ for $1 \leq j \leq d$.
If $\text{value}(\mathcal{P}_j) = \min\{\max \mathcal{P}_j, 1 - \min \mathcal{P}_j\}$ then the claim follows immediately. If $\text{value}(\mathcal{P}_j) = \text{maxwidth}$ then $\max \mathcal{P}_j - \text{maxwidth} \geq 0$ and $\min \mathcal{P}_j + \text{maxwidth} \leq 1$. It follows that $\text{maxwidth} \leq b_j$ and therefore $\text{value}(\mathcal{P}_j) \leq b_j$ for $1 \leq j \leq d$.

(b) Let us now assume that $\text{c-ball}(\Delta) \cap [0,1]^d$ contains no vertex of $[0,1]^d$. Since the center of $\text{c-ball}(\Delta) \notin [0,1]^d$ we can assume that there is some number k, $1 \leq k < d$ such that up to ordering

$$b_1 = \cdots = b_k > b_{k+1} > \cdots > b_d \ .$$

In other words, there is a k-dimensional face $\mathcal{F}$ of $[0,1]^d$ such that the intersection of $\text{c-ball}(\Delta)$ and $\mathcal{F}$ is a k-ball. (If we 'added' the next dimension we would obtain a $(k+1)$-dimensional spherical cap.)
Since $\mathcal{B}$ contains all vertices of the simplex Δ it follows immediately that $b_1 = \cdots = b_k \geq \text{maxwidth}$. Therefore we can conclude that $\text{value}(\mathcal{P}_j) \leq b_j$ for $1 \leq j \leq k$.
Now consider some number $j > k$. Let $\mathcal{F}_{X_j=0}$ be the facet (= $(d-1)$-dimensional face) of $[0,1]^d$ that is contained in the hyperplane $X_j = 0$ and let $\mathcal{F}_{X_j=1}$ be the facet of $[0,1]^d$ that is contained in the hyperplane $X_j = 1$. Since $b_j < b_1$ the box $\mathcal{B}$ 'touches' either $\mathcal{F}_{X_j=1}$ or $\mathcal{F}_{X_j=0}$, i.e., facets of $\mathcal{B}$ are either contained in $\mathcal{F}_{X_j=1}$ or in $\mathcal{F}_{X_j=0}$. And again, since $\mathcal{B}$ also contains all vertices of the simplex Δ it follows that either $\max \mathcal{P}_j \leq b_j$ or that $1 - \min \mathcal{P}_j \leq b_j$. Analogously to case (a) it follows that $\text{maxwidth} \leq b_j$ and therefore $\text{value}(\mathcal{P}_j) \leq b_j$ for $k < j \leq d$.

This concludes the proof of the claim and therefore (4) is shown. By combining (3) and (4) we have proven that

$$d! \cdot \text{vol}\left(\text{c-ball}(\Delta) \cap [0,1]^d\right) \;\geq\; \text{vol}(\mathcal{B}) \;\geq\; \prod_{j=1}^{d} \text{value}(\mathcal{P}_j)$$

which concludes the case (ii) and therefore the proof of lemma 3.

□

Proof of Lemma 5

Lemma 5 *For any value of $a \in [0,1]$ it holds that*

$$\mathbf{Pr}\left[\operatorname{maxwidth}(\mathcal{P}_1, \ldots, \mathcal{P}_d) > \sqrt[d]{a} \quad \textit{and} \quad \prod_{j=1}^{d} \operatorname{value}(\mathcal{P}_j) \le a \right] \;\le\; \mathcal{O}\left(a^d\right) \; .$$

Proof. We will assume some ordering on the sets $\mathcal{P}_1, \ldots, \mathcal{P}_d$ as descibed now. Let the first set $\mathcal{P}_1$ attain the maximal width $\operatorname{maxwidth} > \sqrt[d]{a}$. Furthermore, let the following d_f sets have $\operatorname{value} = \operatorname{maxwidth}$ (the elements lie *far away* from 0 or 1) and let the remaining $d - 1 - d_f =: d_c > 0$ sets have $\operatorname{value} < \operatorname{maxwidth}$ (the elements lie *close* to 0 or 1). So we actually want to consider

$$\begin{aligned}
\mathbf{Pr}\Big[\quad \operatorname{value}(\mathcal{P}_1) \quad &= \quad \operatorname{width}(\mathcal{P}_1) \;=:\; \operatorname{maxwidth} \;>\; \sqrt[d]{a} \\
\operatorname{value}(\mathcal{P}_2) \quad &= \quad \operatorname{maxwidth} \\
&\;\;\vdots \\
\operatorname{value}(\mathcal{P}_{d_f+1}) \quad &= \quad \operatorname{maxwidth} \\
\operatorname{value}(\mathcal{P}_{d_f+2}) \quad &= \quad \min\{\max \mathcal{P}_{d_f+2}, 1 - \min \mathcal{P}_{d_f+2}\} \;<\; \operatorname{maxwidth} \\
&\;\;\vdots \\
\operatorname{value}(\mathcal{P}_{d}) \quad &= \quad \min\{\max \mathcal{P}_{d}, 1 - \min \mathcal{P}_{d}\} \;<\; \operatorname{maxwidth} \\
\text{and} \quad & \quad \prod_{j=1}^{d} \operatorname{value}(\mathcal{P}_j) \;\le\; a \;\Big] \; .
\end{aligned} \tag{5}$$

Since there are at most $d!$ permutations of the sets $\mathcal{P}_1, \ldots, \mathcal{P}_d$ we loose only a factor of at most $d!$ when computing (5) as a bound for the probability desired in the lemma.

In a first step we want now to find an expression for the probability that $\operatorname{maxwidth} = \operatorname{width}(\mathcal{P}_1) > \sqrt[d]{a}$. As in the proof of lemma 4 we fix the two elements in set $\mathcal{P}_1$ that have maximal distance, i.e., $\max \mathcal{P}_1 (= Y)$ and $\min \mathcal{P}_1 (= X)$. Then we can write

$$\mathbf{Pr}\left[\operatorname{width}(\mathcal{P}_1) > \sqrt[d]{a}\right] \;\le\; (d+1) \cdot d \cdot \int_{\sqrt[d]{a}}^{1} \int_{0}^{Y - \sqrt[d]{a}} (Y - X)^{d-1} \, \mathrm{d}X \, \mathrm{d}Y \; . \tag{6}$$

It remains now to tie this probability on the other conditions for the sets $\mathcal{P}_2, \ldots, \mathcal{P}_d$. To do so we will consider the probabilities for these conditions now first seperately.

In a second step we want to consider the probability that the next d_f sets have $\operatorname{value} = \operatorname{maxwidth}$. From the first step it follows that $\operatorname{maxwidth} = (Y - X)$. Furthermore, it suffices to bound the probability that $\operatorname{width} \le (Y - X)$. For $2 \le j \le d_f + 1$ it is

$$\begin{aligned}
\mathbf{Pr}\big[\operatorname{value}(\mathcal{P}_j) = (Y - X)\big] \;&\le\; \mathbf{Pr}\big[\operatorname{width}(\mathcal{P}_j) \le (Y - X)\big] \\
&\le\; (d+1) \cdot (Y - X)^d
\end{aligned} \tag{7}$$

where (7) follows analogously to the proof of lemma 4.

In a third step we want to bound the probability that the sets $\mathcal{P}_{d_f+2}, \ldots, \mathcal{P}_d$ have $\operatorname{value} < (Y - X)$. Furthermore, these sets must not contribute too much to the volume $\prod_{j=1}^{d} \operatorname{value}(\mathcal{P}_j)$ since we want the volume to be at most a. So far we assumed that the sets $\mathcal{P}_1, \ldots, \mathcal{P}_{d_f+1}$ have $\operatorname{value} = (Y - X)$. It follows that already $(Y - X)^{d_f+1}$ is spent on the volume. So for the remaining d_c sets only a volume of

$$\frac{a}{(Y - X)^{d_f+1}} \;=\; \frac{a}{(Y - X)^{d - d_c}} \;=:\; A$$

is left. Therefore our goal is now to compute

$$\mathbf{Pr}\left[d_f + 2 \le j \le d : \operatorname{value}(\mathcal{P}_j) < (Y - X) \quad \text{and} \quad \prod_{j=d_f+2}^{d} \operatorname{value}(\mathcal{P}_j) \le A \right] \; . \tag{8}$$

Since $\mathrm{value}(\mathcal{P}_j) = \min\{\max \mathcal{P}_j, 1 - \min \mathcal{P}_j\}$ it follows that

$$\mathbf{Pr}\Big[d_f + 2 \le j \le d : \mathrm{value}(\mathcal{P}_j) < (Y-X)\Big] \le$$

$$\sum_{I,I'} \mathbf{Pr}\Big[j \in I : \max \mathcal{P}_j < (Y-X) \quad \text{and} \quad j \in I' : 1 - \min \mathcal{P}_j < (Y-X)\Big] \ .$$

where $I \cup I' = \{d_f + 2, \ldots, d\}$. We conslude that

$$(8) \quad \le \quad 2^{d_c} \cdot \mathbf{Pr}\Big[d_f + 2 \le j \le d : \max \mathcal{P}_j \le (Y-X) \quad \text{and} \quad \prod_{j=d_f+2}^{d} \max \mathcal{P}_j \le A\Big]$$

and thus we will show now the following claim in order to continue with the proof of lemma 5.

Claim: For any value of $a \in [0,1]$ *it holds that*

$$\mathbf{Pr}\Big[d_f + 2 \le j \le d : \max \mathcal{P}_j \le (Y-X) \quad \text{and} \quad \prod_{j=d_f+2}^{d} \max \mathcal{P}_j \le A\Big]$$
$$\le \quad \mathcal{O}\left(A^d \cdot (Y-X)^{d_c}\right) \ .$$

Proof of Claim: We present the following integral

$$d^{d_c - 1} \cdot \int_0^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \left(\min\left\{(Y-X), \frac{A}{Z_1 \cdots Z_{d_c-1}}\right\}\right)^{d+1} \mathrm{d}Z_1 \cdots \mathrm{d}Z_{d_c-1} \tag{9}$$

and will now explain its meaning.

As before we fixed for each of the $d_c - 1$ sets $\mathcal{P}_{d_f+2}, \ldots, \mathcal{P}_{d-1}$ the maximal element ($= Z_1, \ldots, Z_{d_c-1}$) which gives the factor d^{d_c-1} in front of the whole integral. The $d_c - 1$ integrals give the range of the maximal elements, namely from 0 to $(Y-X)$. Now consider a set $\mathcal{P}_j$. For the remaining d elements of $\mathcal{P}_j$ the term Z_j^d gives the probability that they are not greater than the maximal element Z_j. Thus we made sure that for $d_f + 2 \le j \le d - 1$ the condition $\max \mathcal{P}_j \le (Y-X)$ is fullfilled.

For the last set $\mathcal{P}_d$ we have to assure on the one hand that all $d+1$ elements are not greater than $(Y-X)$ and on the other hand that the volume bound is not exceeded.

The probability for the first condition is exactly $(Y-X)^{d+1}$. For the second condition observe that the sets $\mathcal{P}_{d_f+1}, \ldots, \mathcal{P}_{d-1}$ spent already $Z_1 \cdots Z_{d_c-1}$ on the volume. Therefore, set $\mathcal{P}_d$ may still spend at most $A/(Z_1 \cdots Z_{d_c-1})$ on the volume. So all $d+1$ elements in $\mathcal{P}_d$ must not exceed this bound and the probability for this is $(A/(Z_1 \cdots Z_{d_c-1}))^{d+1}$.

So the integral (9) expresses the probability demanded in the claim and if it is shown that $(9) \le \mathcal{O}\left(A^d \cdot (Y-X)^{d_c}\right)$ also the claim is shown.

In order to solve the integral in (9) we start with some preliminary observation. For all $k \in \mathbb{N}$ let

$$\mathcal{R}(k) := \frac{A}{(Y-X)^k} \ .$$

Since $A = \frac{a}{(Y-X)^{d-d_c}}$ and $(Y-X) > \sqrt[d]{a}$ it follows:

Observation 2 *It holds that* $\qquad (Y-X) > \mathcal{R}(d_c - 1).$

In a first step we will split up the outermost integral in (9) into two integrals, one going from 0 to $\mathcal{R}(d_c - 1)$ and the other from $\mathcal{R}(d_c - 1)$ to $(Y-X)$. We will see that the first integral can be solved in a straightforward way. For the second integral we apply again our strategy but this time we split up the second outermost integral (though we use some slightly modified integration boundaries). This results in an interative process leading to a final solution for the integral in (9).

For better readability let us also introduce the following abbreviation, let

$$\mathcal{K} \;=\; \mathcal{K}(Z_1, \ldots, Z_{d_c-1}) \;:=\; \left(\min\left\{(Y-X), \frac{A}{Z_1 \cdots Z_{d_c-1}}\right\}\right)^{d+1} \quad .$$

Let us now have a closer look on the outermost integral in (9). It is

$$\begin{aligned}
&\int_0^{(Y-X)} \int_0^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \mathcal{K} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \\
= \;& \int_0^{\mathcal{R}(d_c-1)} \int_0^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \mathcal{K} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \qquad (10) \\
+ \;& \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_0^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \mathcal{K} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \quad . \qquad (11)
\end{aligned}$$

In integral (10) the variable Z_{d_c-1} is bounded by $\mathcal{R}(d_c-1)$ and the variables $Z_1, \ldots, Z_{d_c-2}$ are bounded by $(Y-X)$. Therefore, we can conclude that

$$\frac{A}{Z_1 \cdots Z_{d_c-1}} \;\geq\; \frac{A}{(Y-X)^{d_c-2} \cdot \mathcal{R}(d_c-1)} \;=\; (Y-X) \quad .$$

It follows that $\mathcal{K} = (Y-X)^{d+1}$ and therefore

$$\begin{aligned}
(10) \;&=\; \int_0^{\mathcal{R}(d_c-1)} \int_0^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot (Y-X)^{d+1} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \\
&=\; \int_0^{\mathcal{R}(d_c-1)} \left(\frac{1}{d+1}\right)^{d_c-2} \cdot (Y-X)^{(d+1)\cdot(d_c-1)} \cdot Z_{d_c-1}^d \,\mathrm{d}Z_{d_c-1} \\
&=\; \left(\frac{1}{d+1}\right)^{d_c-1} \cdot (Y-X)^{(d+1)\cdot(d_c-1)} \cdot A^{d+1} \cdot \left(\frac{1}{(Y-X)}\right)^{(d+1)\cdot(d_c-1)} \\
&=\; \left(\frac{1}{d+1}\right)^{d_c-1} \cdot A^{d+1}
\end{aligned}$$

In order to solve integral (11) we will split up the second outermost integral into two integrals, one going from 0 to $\mathcal{R}(d_c-2) \cdot Z_{d_c-1}^{-1}$ and the other from $\mathcal{R}(d_c-2) \cdot Z_{d_c-1}^{-1}$ to $(Y-X)$.

$$\begin{aligned}
(11) \;=\;& \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_0^{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \mathcal{K} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \qquad (12) \\
+\;& \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}}^{(Y-X)} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \cdot \mathcal{K} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \qquad (13)
\end{aligned}$$

The integral (12) can also be solved in a straightforward way. Since variable Z_{d_c-2} is bounded by $\mathcal{R}(d_c-2) \cdot Z_{d_c-1}^{-1}$ we can conclude that

$$\frac{A}{Z_1 \cdots Z_{d_c-1}} \;\geq\; \frac{A \cdot Z_{d_c-1}}{(Y-X)^{d_c-3} \cdot \mathcal{R}(d_c-2) \cdot Z_{d_c-1}} \;=\; (Y-X) \quad .$$

It follows that $\mathcal{K} = (Y-X)^{d+1}$ and therefore

$$\begin{aligned}
(12) \;&=\; \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_0^{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}} \cdots \int_0^{(Y-X)} Z_1^d \cdots Z_{d_c-1}^d \\
&\qquad\qquad \cdot (Y-X)^{d+1} \,\mathrm{d}Z_1 \cdots \,\mathrm{d}Z_{d_c-1} \\
&=\; \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_0^{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}} \left(\frac{1}{d+1}\right)^{d_c-3} \cdot Z_{d_c-2}^d \cdot Z_{d_c-1}^d \\
&\qquad\qquad \cdot (Y-X)^{(d+1)\cdot(d_c-2)} \,\mathrm{d}Z_{d_c-2} \,\mathrm{d}Z_{d_c-1} \\
&=\; \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \left(\frac{1}{d+1}\right)^{d_c-2} \cdot A^{d+1} \cdot \frac{1}{Z_{d_c-1}} \,\mathrm{d}Z_{d_c-1} \\
&=\; \left(\frac{1}{d+1}\right)^{d_c-2} \cdot A^{d+1} \cdot \ln\left(\frac{(Y-X)^{d_c}}{A}\right)
\end{aligned}$$

In order to solve integral (13) we will split up the third outermost integral into two integrals, one going from 0 to $\mathcal{R}(d_c-3)\cdot Z_{d_c-2}^{-1}\cdot Z_{d_c-1}^{-1}$ and the other from $\mathcal{R}_{d_c-3}\cdot Z_{d_c-2}^{-1}\cdot Z_{d_c-1}^{-1}$ to $(Y-X)$. The whole process continues now analogously. But still we will carry out here the computation of the next integral for reasons of better illustration although the depiction becomes more and more uncomfortable. It is

$$
\begin{aligned}
(13) \quad = \quad & \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}}^{(Y-X)} \int_{0}^{\mathcal{R}(d_c-3)\cdot Z_{d_c-2}^{-1}\cdot Z_{d_c-1}^{-1}} \cdots \int_{0}^{(Y-X)} \quad \cdots & (14)\\
+ \quad & \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}}^{(Y-X)} \int_{\mathcal{R}(d_c-3)\cdot Z_{d_c-2}^{-1}\cdot Z_{d_c-1}^{-1}}^{(Y-X)} \cdots \int_{0}^{(Y-X)} \quad \cdots & (15)
\end{aligned}
$$

As before we can conclude from the ranges of the variables $Z_1,\ldots,Z_{d_c-1}$ that $\mathcal{K}=(Y-X)^{d+1}$. After solving the inner integrals up to the two outermost it remains that

$$
\begin{aligned}
(14) \quad &= \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \int_{\mathcal{R}(d_c-2)\cdot Z_{d_c-1}^{-1}}^{(Y-X)} \left(\frac{1}{d+1}\right)^{d_c-3} \cdot A^{d+1} \cdot \frac{1}{Z_{d_c-2}\cdot Z_{d_c-1}} \,\mathrm{d}Z_{d_c-2}\,\mathrm{d}Z_{d_c-1}\\
&= \left(\frac{1}{d+1}\right)^{d_c-3} \cdot A^{d+1} \cdot \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \ln\left(Z_{d_c-1}\cdot\frac{(Y-X)^{d_c-1}}{A}\right) \cdot \frac{1}{Z_{d_c-1}} \,\mathrm{d}Z_{d_c-1}\\
&\leq \left(\frac{1}{d+1}\right)^{d_c-3} \cdot A^{d+1} \cdot \ln\left(\frac{(Y-X)^{d_c}}{A}\right) \cdot \int_{\mathcal{R}(d_c-1)}^{(Y-X)} \frac{1}{Z_{d_c-1}} \,\mathrm{d}Z_{d_c-1}\\
&= \left(\frac{1}{d+1}\right)^{d_c-3} \cdot A^{d+1} \cdot \ln\left(\frac{(Y-X)^{d_c}}{A}\right)^2
\end{aligned}
$$

Of course with integral (15) we proceed analogously. To summarize the results so far we conclude that

$$
(9) \quad \leq \quad d^{d_c-1}\cdot A^{d+1}\cdot\sum_{i=1}^{d_c}\left(\frac{1}{d+1}\right)^{d_c-i}\cdot\ln\Bigg(\underbrace{\frac{(Y-X)^{d_c}}{A}}_{>1}\Bigg)^{i-1} \quad . \qquad (16)
$$

Since for every number k there is a constant c_k s.t. $c_k\cdot x \geq \ln(x)^k \quad \forall\, x>1$ we get that

$$
\begin{aligned}
(16) \quad &\leq \quad d^{d_c-1}\cdot\sum_{i=1}^{d_c}\left(\frac{1}{d+1}\right)^{d_c-i}\cdot A^{d+1}\cdot c_{i-1}\cdot\frac{(Y-X)^{d_c}}{A}\\
&\leq \quad d^{d_c-1}\cdot A^{d}\cdot(Y-X)^{d_c}\cdot c_{d_c-1}\cdot\sum_{i=0}^{d_c-1}\left(\frac{1}{d+1}\right)^{i}\\
&\leq \quad d^{d_c-2}\cdot c_{d_c}\cdot A^{d}\cdot(Y-X)^{d_c-1}
\end{aligned}
$$

and thus the claim is shown.

It follows that

$$
\begin{aligned}
&\mathbf{Pr}\left[d_f+2\leq j\leq d:\mathrm{value}(\mathcal{P}_j)<(Y-X) \quad \text{and} \quad \prod_{j=d_f+2}^{d}\mathrm{value}(\mathcal{P}_j)\leq A\right]\\
&\qquad\qquad \leq 2^{d_c}\cdot d^{d_c-2}\cdot c_{d_c}\cdot A^{d}\cdot(Y-X)^{d_c-1} \quad . \qquad (17)
\end{aligned}
$$

We can now put all the pieces together and present an overall expression for the probability (5) in order to show lemma

5. For this purpose we extend the integral in (6) in the following way. It is

$$\begin{aligned}
(5) \quad &= \quad (d+1) \cdot d \cdot \int_{\sqrt[d]{a}}^{1} \int_{0}^{Y-\sqrt[d]{a}} (Y-X)^{d-1} \cdot \prod_{j=2}^{d_f+1} \mathbf{Pr}\big[\,\text{value}(\mathcal{P}_j) = (Y-X)\big] \cdot \\
&\qquad \mathbf{Pr}\Big[d_f + 2 \leq j \leq d : \text{value}(\mathcal{P}_j) < (Y-X) \quad \text{and} \prod_{j=d_f+2}^{d} \text{value}(\mathcal{P}_j) \leq A\Big] \;\mathrm{d}X \;\mathrm{d}Y \\
&\leq \quad (d+1) \cdot d \cdot \int_{\sqrt[d]{a}}^{1} \int_{0}^{Y-\sqrt[d]{a}} (Y-X)^{d-1} \cdot \Big((d+1) \cdot (Y-X)^{d}\Big)^{d_f} \cdot \\
&\qquad \Big(2^{d_c} \cdot d^{d_c-2} \cdot c_{d_c-1} \cdot A^{d} \cdot (Y-X)^{d_c}\Big) \;\mathrm{d}X \;\mathrm{d}Y \\
&= \quad (d+1)^{d_f+1} \cdot 2^{d_c} \cdot c_{d_c-1} \cdot a^{d} \cdot \int_{\sqrt[d]{a}}^{1} \int_{0}^{Y-\sqrt[d]{a}} (Y-X)^{d_c-1} \;\mathrm{d}X \;\mathrm{d}Y \;\; .
\end{aligned}$$

The second last step follows with (7) and (17) and the last step since $A = \frac{a}{(Y-X)^{d-d_c}}$ and $d_f = d - d_c - 1$.

Now for the integral in this last expression we get

$$\begin{aligned}
\int_{\sqrt[d]{a}}^{1} \int_{0}^{Y-\sqrt[d]{a}} (Y-X)^{d_c-1} \;\mathrm{d}X \;\mathrm{d}Y \;&=\; \frac{1}{d_c} \cdot \int_{\sqrt[d]{a}}^{1} Y^{d_c} - \big(\sqrt[d]{a}\big)^{d_c} \;\mathrm{d}Y \\
= \quad \frac{1}{d_c} \cdot \left(\frac{1}{d_c+1} - \big(\sqrt[d]{a}\big)^{d_c} \cdot \underbrace{\left(1 - \left(1 - \frac{1}{d_c+1}\right) \cdot \sqrt[d]{a}\right)}_{>0}\right) \;&\leq\; \frac{1}{d_c \cdot (d_c+1)} \quad .
\end{aligned}$$

Thus we can finally conclude that

$$(5) \;\leq\; (d+1)^{d_f+1} \cdot 2^{d_c} \cdot c_{d_c-1} \cdot \frac{1}{d_c \cdot (d_c+1)} \cdot a^{d} \;\leq\; \mathcal{O}(a^{d})$$

which ends the proof of lemma 5.

□