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Computable operators on regular sets

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For regular sets in Euclidean space, previous work has identified twelve 'basic' computability notions to (pairs of) which many previous notions considered in literature were shown to be equivalent. With respect to those basic notions we now investigate on the computability of natural operations on regular sets: union, intersection, complement, convex hull, image, and pre-image under suitable classes of functions. It turns out that only few of these notions are suitable in the sense of rendering all those operations uniformly computable.

1 Introduction

Regular subsets of Euclidean space $\mathbb{R}^d$ form an important sub-class

$$\mathfrak{R}^d = \{R \subseteq \mathbb{R}^d : \overline{R^\circ} = R\}$$

of the closed ones. Their applications to solid modeling come from the property of physical bodies to be 'solid' (i. e., full-dimensional although not necessarily convex) which is precisely reflected by the requirement upon R to coincide with the topological closure of its interior $\overline{R^\circ}$.

Do two mechanical components touch or can they move freely past each other ? Questions like this illustrate that numerical reliability may be crucial to CAD/CAE-designs and -tools. Consequently it was only natural to start investigating on computability issues related to regular sets in a realistic, i. e., approximate rather than algebraic model of real number computation [3]. In fact, a variety of generalizations and extensions of Turing's seminal notion of computable real numbers [11] to certain subsets of $\mathbb{R}^d$ have been considered (mostly independently) in literature. [13] systematically compared these notions for the case of regular subsets, proved them to be equivalent to (pairs of) certain 'basic' ones, and revealed the *weaker/stronger* relations among the latter.

Having in this sense settled the computability problem for regular sets themselves, we now ask for effectivity of *operators* on them. Is, for instance, union

$$\cup : \mathfrak{R}^d \times \mathfrak{R}^d \ni (R_1, R_2) \longmapsto R_1 \cup R_2 \in \mathfrak{R}^d$$

a computable mapping and, if so, with respect to which ones of the basic encodings considered in [13] ? Similar questions arise naturally for intersection, complement, convex hull, image $R \longmapsto f[R]$, and pre-image $R \longmapsto g^{-1}[R]$. But before addressing computability of these operations, let us first assert them to indeed map regular sets to regular sets.

Proposition 1.1

a) *Let* $R_1, R_2 \in \mathfrak{R}^d$. *Then* $R_1 \cup R_2 \in \mathfrak{R}^d$.

b) *Let* $R_i \in \mathfrak{R}^d$ *for each* $i \in I$; *then* $\overline{\bigcup_{i\in I} R_i} \in \mathfrak{R}^d$.

c) *Let* $R \in \mathfrak{R}^d$. *Then* $\overline{\mathbb{R}^d \setminus R} \in \mathfrak{R}^d$.

d) *Let* $R \in \mathfrak{R}^m$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *continuous and open* (*mapping open sets to open sets*); *then* $f^{-1}[R] \in \mathfrak{R}^n$.

e) *Let* $R \in \mathfrak{R}^n$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *continuous, open, and closed. Then* $g[R] \in \mathfrak{R}^m$.

f) *Let* $R \subseteq \mathbb{R}^n$ *be regular and bounded,* $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *continuous and open; then* $g[R] \subseteq \mathbb{R}^m$ *is also regular and bounded.*

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The intersection of two regular sets is in general not regular. The requirements imposed on f and g are sharp:

- *Function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto t^2$, is continuous and $R := [-1, 0]$ is regular but $f^{-1}[R] = \{0\}$ is not.*
- *Function $g : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto 0$, is continuous and closed but $g[R] = \{0\}$ is not regular for any regular set R.*
- *Function $g : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto \exp(t)$, is continuous and open, $R := (-\infty, 0]$ is regular but $g[R] = (0, 1]$ is not.*

P r o o f. Recall (e. g., Lemma 4.2l)) that a set is regular iff it is the closure of *some* open set.

a) follows from Lemma 4.2g): For $R_1 = \overline{U_1}$ and $R_2 = \overline{U_2}$, $R_1 \cup R_2 = \overline{U_1 \cup U_2}$, the closure of an openset.

b) Let $R_i = \overline{U_i}$, $R := \overline{\bigcup R_i}$, $U := \bigcup U_i$ open. Then obviously $U \subseteq \bigcup \overline{U_i}$ and $\bigcup \overline{U_i} \subseteq \overline{U}$ by virtue of Lemma 4.2f); hence $\overline{U} = \overline{\bigcup \overline{U_i}} = R$.

c) $\mathbb{R}^d \setminus R$ is the complement of a closed, i. e., an open set.

d) Let $R = \overline{U}$. Then $V := g^{-1}[U]$ is open because of continuity. Furthermore, $g^{-1}[R] = \overline{V}$ according to Lemmas 4.4a) and 4.4b).

e) Let $R = \overline{U}$. Then $V := f[U]$ is open since f is. Moreover, $f[R] = \overline{V}$ according to Lemmas 4.4a) and 4.4c).

f) Similarly by means of Lemma 4.4d).

$R_1' := [-1, 0] \cup [2, 3]$ and $R_2' := [0, 1] \cup [2, 3]$ are both regular and even compact, yet their intersection – although having even non-empty interior – is not regular. □

The domains on which to consider the above set operations are thus laid down. Section 3 states our according computability results: first those concerning union, intersection, closed complement; then, in Subsection 3.1, thosefor image and pre-image under suitable classes of functions; and Subsection 3.2 finally investigates on convex hull computation. Section 4 collects some technical toolswhich we did not want to interrupt the train of thoughts in the main part with. Section 5 concludes with a comparing discussion of the considered encodings. One pair of them is identified to have the particular feature of rendering *all* of the above operations uniformly computable.

2 Computability

The present section recalls the basic notions for computability of regular sets from [13]. In fact each such notion is a *representation*, that is, a naming system of regular sets assigning each $R \in \mathfrak{R}^d$ one or more infinite binary strings $\bar{\sigma} \in \{0, 1\}^\omega$. A well-established framework for such encodings called TTE (Type-2 Theory of Effectivity) shall be used in the sequel as a convenient language to formulate and prove in a modular way our computability results; for details on TTE refer to [12].

Our focus shall lie on *uniform* computability, that is, the question whether, for a (say, unary) operation on regular sets, there exists a Type-2 Machine which, upon input of (an encoding of) a regular set R, outputs (an encoding of) its image R' under the mentioned operation under consideration. This includes non-uniform computability where for each computable regular set R, its image R' has to be computable again. Although here the transformation $R \longrightarrow R'$ itself need not necessarily be effective in general, uniform *non*-computability quite often does conversely imply non-uniform non-computability [2].

Definition 2.1 Let $\mathfrak{O}^d$ denote the family of open subsets of $\mathbb{R}^d$; similarly $\mathfrak{A}^d$ for the closed ones. $\theta^d_<$ is the following representation for $\mathfrak{O}^d$: a name for U is a list of (encodings of) rational centers and radii of open Euclidean balls covering exactly U. $\psi^d_<$ is the following representation for $\mathfrak{A}^d$: a name for A is a sequence of (encodings of) real vectors dense in A. A $\theta^d_>$-name for U is a $\psi^d_<$-name for $\mathbb{R}^d \setminus U$; a $\psi^d_>$-name for A is a $\theta^d_<$-name for $\mathbb{R}^d \setminus A$.

Since a regular set R can be regarded (cf. [13, Lemma 3.2b]) as a certain closed set R, or as the closure of some open set $\overline{U}$, or as the closure of the interior of some closed set $\overline{A^\circ}$, [13] arrives at the following representations for $\mathfrak{R}^d$:

- $\psi^d_>$ according to Definition 2.1
- $\theta^{\mathfrak{R}d}_<$: a name for R is a $\theta^d_<$-name for R°;

- $\overline{\psi^{\circ d}_{>}}$: a name for R is a $\psi^d_>$-name for some $A \in \mathfrak{A}^d$ such that $\overline{A^\circ} = R$;
- $\overline{\theta}^d_<$: a name for R is a $\theta^d_<$-name for some $U \in \mathfrak{O}^d$ such that $\overline{U} = R$;
- $\overset{\circ}{\theta}{}^d_>$: a name for R is a $\theta^d_>$-name for R°;
- $\psi^d_<$ according to Definition 2.1
- $\overline{\theta}^d_>$: a name for R is a $\theta^d_>$-name for some $U \in \mathfrak{O}^d$ such that $\overline{U} = R$;
- $\overline{\psi^{\circ d}_{<}}$: a name for R is a $\psi^d_<$-name for some $A \in \mathfrak{A}^d$ such that $\overline{A^\circ} = R$;
- $\vartheta^d_<$: an enumeration of (the encodings) of all rational interior points, i. e., of $\mathbb{Q}^d \cap R^\circ$;
- $\vartheta^d_>$: an enumeration of (the encodings) of all rational exterior points, i. e., of $\mathbb{Q}^d \setminus R$;
- $\overline{\vartheta}^d_<$: an enumeration of some $Q \subseteq \mathbb{Q}^d$ such that $\overline{Q} = R$;
- $\overline{\vartheta}^d_>$: an enumeration of some $Q \subseteq \mathbb{Q}^d$ such that $\overline{Q} = \overline{\mathbb{R}^d \setminus R}$;
- $\overline{\vartheta^{\circ d}_<}$: an enumeration of some $Q \subseteq \mathbb{Q}^d$ such that $\overline{\left(\overline{Q}\right)^\circ} = R$;
- $\overline{\vartheta^{\circ d}_>}$: an enumeration of some $Q \subseteq \mathbb{Q}^d$ such that $\overline{\left(\overline{Q}\right)^\circ} = \overline{\mathbb{R}^d \setminus R}$.

Theorem 4.9 of [13] proved many other representations considered in literature to be equivalent to the *join* of pairs of the above. It thus suffices to treat these 'basic' ones and in fact, [13, Theorem 5.2] revealed the relations among them as compiled into the following figure:

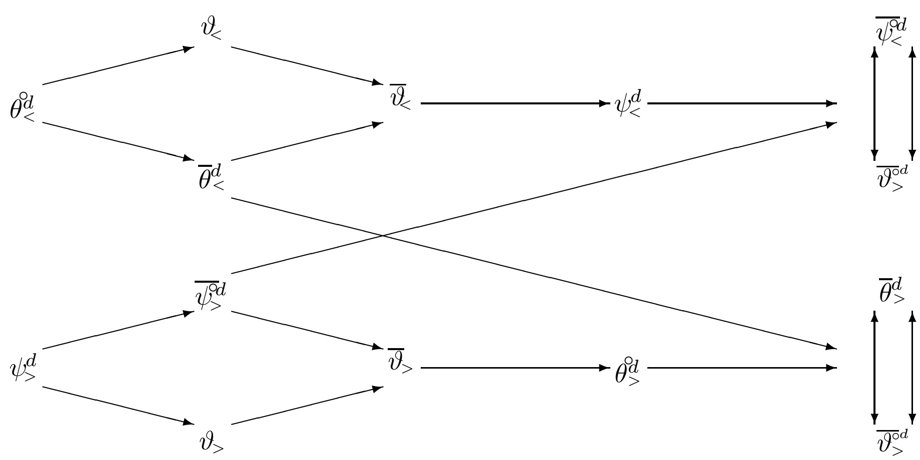

Figure 1: Computability relations among basic representations

In Figure 1, directed arrows (+transitivity) indicate uniform convertability of according names whereas absence of a directed path means computable independence. Observe that $\overline{\vartheta^{\circ d}_>}$ and $\overline{\theta}^d_>$ on the one hand and $\overline{\vartheta^{\circ d}_<}$ and $\overline{\psi^{\circ d}_<}$ on the other hand are equivalent, respectively, whereas the others exhibit pairwise inequivalence. Representations with index "$<$" will be called *positive* whereas those with index "$>$" are *negative* in a sense explained in [13].

One thus has essentially twelve distinct basic representations. So with respect to which ones of them are the operations considered in Proposition 1.1 computable ?

3 Results

We begin with computability of the three Boolean operations: closed complement, union, and intersection. Already [7] contains remarks (without proofs) about effectivity of the latter two with respect to some of the representations from Definition 2.1. The present work however goes for a complete systematic treatment in Theorems 3.2 and 3.4. But let us start with the following theorem.

Theorem 3.1 *Closed complement of regular sets, i. e., the mapping* $\mathfrak{R}^d \ni R \longmapsto \overline{\mathbb{R}^d \setminus R} = \mathbb{R}^d \setminus \mathring{R} \in \mathfrak{R}^d$ *is*

$(\mathring{\theta}^d_< \to \psi^d_>)$*-computable;* $(\vartheta^d_< \to \vartheta^d_>)$*-computable;* $(\overline{\theta}^d_< \to \overline{\mathring{\psi}}^d_>)$*-computable;*
$(\overline{\vartheta}^d_< \to \overline{\vartheta}^d_>)$*-computable;* $(\psi^d_< \to \mathring{\theta}^d_>)$*-computable;* $(\overline{\mathring{\psi}}^d_< \to \overline{\theta}^d_>)$*-computable;*

$(\vartheta^d_> \to \vartheta^d_<)$*-computable;* $(\psi^d_> \to \mathring{\theta}^d_<)$*-computable;* $(\overline{\mathring{\psi}}^d_> \to \overline{\theta}^d_<)$*-computable;*
$(\overline{\vartheta}^d_> \to \overline{\vartheta}^d_<)$*-computable;* $(\mathring{\theta}^d_> \to \psi^d_<)$*-computable;* $(\overline{\theta}^d_> \to \overline{\mathring{\psi}}^d_<)$*-computable;*

$(\overline{\theta}^d_< \to \overline{\mathring{\psi}}^d_<)$*-computable;* $(\overline{\mathring{\psi}}^d_> \to \overline{\theta}^d_>)$*-computable.*

Observe that, different from the others, the last line of claims relates representations of *same* 'sign'; namely, given strong positive (index $_<$) information on R, one *can* deduce (weak) positive information on R's closed complement.

The proof of Theorem 3.1 is immediate regarding [13, Observation 3.5] and taking into account Figure 1.

Obviously, the representation for input may always be replaced by any stronger one and for output by any weaker one and still retain computability. But in fact, Theorem 3.1 is optimal in the sense that any weaker input or stronger output representation according to Figure 1 renders the claims incorrect: this follows from closed complement being an involution, i. e., self-inverse: $\overline{\mathbb{R}^d \setminus \overline{\mathbb{R}^d \setminus R}} = R$.

For the remaining operations on regular sets considered in this work – union, intersection, pre-image, image, and convex hull, that is – we will mostly focus on input and output being encoded in the same way[1]. Indeed by virtue of effective composition, results asserting, say, $(\gamma \times \gamma \to \gamma)$-computability of these operations can easily be used as building blocks to more complicated problems involving regular sets; cf., e. g., Corollary 5.1.

Theorem 3.2 *Binary union of* d*-dimensional regular sets, i. e., the mapping*

$$\cup : \mathfrak{R}^d \times \mathfrak{R}^d \ni (R_1, R_2) \longmapsto R_1 \cup R_2 \in \mathfrak{R}^d ,$$

is $(\gamma \times \gamma \to \gamma)$*-computable (indicated by "*$+$*") and* $(\gamma \times \gamma \to \gamma)$*-discontinuous ("*$-$*") for the following choices:*

γ	$\mathring{\theta}^d_<$	$\overline{\theta}^d_<$	$\vartheta^d_<$	$\overline{\vartheta}^d_<$	$\psi^d_<$	$\overline{\mathring{\psi}}^d_<$	$\psi^d_>$	$\overline{\mathring{\psi}}^d_>$	$\vartheta^d_>$	$\overline{\vartheta}^d_>$	$\mathring{\theta}^d_>$	$\overline{\theta}^d_>$
	$-$	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$+$

The affirmative results above also hold for n*-ary union.*

The negative claims are proven by simple counter-examples whereas positive claims mostly rely on known results concerning computability of union/intersection of open and closed sets ([12, Theorem 5.1.13 and Corollary 5.1.18]) combined with commutation properties of set operations "$\cup$", topological closure, and topological interior, collected in Lemmas 4.2 and 4.3 of the present work.

P r o o f . By induction, it suffices to consider only binary union. Furthermore we choose for some of the twelve claims to prove stronger versions like, e. g., $(\mathring{\theta}^d_< \times \mathring{\theta}^d_< \to \vartheta^d_<)$-discontinuity instead of both $(\mathring{\theta}^d_< \times \mathring{\theta}^d_< \to \mathring{\theta}^d_<)$-discontinuity and $(\vartheta^d_< \times \vartheta^d_< \to \vartheta^d_<)$-discontinuity.

$\mathring{\theta}^d_< \times \mathring{\theta}^d_< \not\to \vartheta^d_<$. Consider for $d = 1$ the intervals $R_1 = [-1, 0]$, $R_2 = [0, 1]$, and $\theta_<$ given by sequences of open rational intervals $(-1, -\frac{1}{n})$ and $(\frac{1}{n}, 1)$, respectively. Were union $\vartheta^d_<$-continuous upon this input, then one could assert that $\boldsymbol{q} := 0 \in (R_1 \cup R_2)^\circ$ knowing only finitely many of the above intervals. But for finite n, the according initial segments could belong as well to $R'_1 := [-1, -\frac{1}{n}]$ and $R'_2 := [\frac{1}{n}, 1]$ with $\boldsymbol{q} \notin (R'_1 \cup R'_2)^\circ$: contradiction.

$\overline{\theta}^d_< \times \overline{\theta}^d_< \to \overline{\theta}^d_<$. Let open $U_1, U_2 \subseteq \mathbb{R}^d$ be given by respective $\theta^d_<$-names such that $\overline{U_1} = R_1$, $\overline{U_2} = R_2$. Compute by virtue of [12,Corollary 5.1.18.2] a $\theta^d_<$-name for open $U := U_1 \cup U_2$. Then $\overline{U} = R := R_1 \cup R_2$ according to Lemma 4.2g).

$\overline{\vartheta}^d_< \times \overline{\vartheta}^d_< \to \overline{\vartheta}^d_<$. Given enumerations $Q_1, Q_2 \subseteq \mathbb{Q}^d$ such that $\overline{Q_i} = R_i$, merge them, that is, compute $Q := Q_1 \cup Q_2$. Then, again by means of Lemma 4.2g), $\overline{Q} = R_1 \cup R_2$.

[1] However see Remark 3.3.

$\psi^d_< \times \psi^d_< \to \psi^d_<$. [12,Theorem 5.1.13.1].

$\overline{\psi}^{\circ d}_< \times \overline{\psi}^{\circ d}_< \to \overline{\psi}^{\circ d}_<$. Let closed $A_1,\ A_2$ be given by respective $\psi^d_<$-names such that $R_i = \overline{A_i^\circ}$. Again exploit [12, Theorem 5.1.13.1] to $\psi^d_<$-compute $A := A_1 \cup A_2$. Then $\overline{A^\circ} = R_1 \cup R_2$ holds due to Lemma 4.3b).

$\psi^d_> \times \psi^d_> \to \psi^d_>$. [12, Theorem 5.1.13.1].

$\overline{\psi}^{\circ d}_> \times \overline{\psi}^{\circ d}_> \to \overline{\psi}^{\circ d}_>$. Similarly to $(\overline{\psi}^{\circ d}_< \times \overline{\psi}^{\circ d}_< \to \overline{\psi}^{\circ d}_<)$-computability.

$\vartheta^d_> \times \vartheta^d_> \to \vartheta^d_>$. Let $Q_i = \mathbb{Q}^d \setminus R_i$ denote the sets enumerated by the respective given $\vartheta^d_>$-names of R_i. Check through all pairs $(q_1, q_2) \in Q_1 \times Q_2$ and report whenever $q_1 = q_2$: this yields output $Q := Q_1 \cap Q_2$. Now observe that $Q = \mathbb{Q}^d \setminus (R_1 \cup R_2)$.

$\overline{\vartheta}^d_> \times \overline{\vartheta}^d_> \not\to \mathring{\theta}^d_>$. Similarly to $(\overline{\vartheta}^d_< \times \overline{\vartheta}^d_< \to \psi^d_<)$-discontinuity of binary intersection proved below.

$\overline{\theta}^d_> \times \overline{\theta}^d_> \to \overline{\theta}^d_>$. Similarly to $(\overline{\psi}^{\circ d}_< \times \overline{\psi}^{\circ d}_< \to \overline{\psi}^{\circ d}_<)$-computability of binary intersection proved below. □

Remark 3.3 Concerning the negative results of Theorem 3.2 it is natural to ask (and thus temporarily lift the "*same input as output encoding*" restriction) whether supply of stronger information might render these cases computable as well. For $\vartheta^d_<$, for instance, this has already been revealed to not be the case in the first item of the above proof. Regarding $\overline{\vartheta}^d_>$- and $\mathring{\theta}^d_>$-discontinuity on the other hand one can show (by arguments similar to those presented below for binary intersection) that binary union *is*

- $(\mathring{\theta}^d_> \times \overline{\psi}^{\circ d}_> \to \mathring{\theta}^d_>)$-computable (proven similarly to $(\psi^d_< \times \overline{\theta}^d_< \to \psi^d_<)$-computability of binary intersection);
- $(\overline{\vartheta}^d_> \times \overline{\psi}^{\circ d}_> \to \overline{\vartheta}^d_>)$-computable (proven similarly to $(\overline{\vartheta}^d_< \times \overline{\theta}^d_< \to \overline{\vartheta}^d_<)$-computability of binary intersection);
- $(\overline{\vartheta}^d_> \times \vartheta^d_> \to \overline{\vartheta}^d_>)$-computable (proven similarly to $(\overline{\vartheta}^d_< \times \vartheta^d_< \to \overline{\vartheta}^d_<)$-computability of binary intersection).

Now turning to intersection, one has to pay attention because this operation is only partial since $R_1 \cap R_2$ need not be regular for $R_1,\ R_2 \in \mathfrak{R}^d$; cf. last line of Proposition 1.1.

Theorem 3.4 *Binary intersection of d-dimensional regular sets, i. e., the partial mapping*

$$\cap\ :\ (R_1, R_2) \longmapsto R_1 \cap R_2 \in \mathfrak{R}^d, \quad \mathrm{dom}(\cap) := \{(R_1, R_2) : R_1,\ R_2,\ R_1 \cap R_2 \in \mathfrak{R}^d\}$$

is $(\gamma \times \gamma \to \gamma)$-computable ("+") *and $(\gamma \times \gamma \to \gamma)$-discontinuous* ("−") *for the following choices:*

γ	$\mathring{\theta}^d_<$	$\overline{\theta}^d_<$	$\vartheta^d_<$	$\overline{\vartheta}^d_<$	$\psi^d_<$	$\overline{\psi}^{\circ d}_<$	$\psi^d_>$	$\overline{\psi}^{\circ d}_>$	$\vartheta^d_>$	$\overline{\vartheta}^d_>$	$\mathring{\theta}^d_>$	$\overline{\theta}^d_>$
	+	+	+	−	−	+	+	+	+	+	+	+

The affirmative claims above also hold for n-ary intersection.

Observe that here, computability of the n-ary case does *not* follow inductively from the binary one since for the intermediate results regularity cannot in general be asserted:

$$n = 3,\ d = 1, \quad R_1 = [1, 2] \cup [8, 9],\ \ R_2 = [2, 3] \cup [8, 9],\ \ R_3 = [0, 1] \cup [3, 4] \cup [8, 9]\,.$$

We also point out the broken de Morgan symmetry caused by intersection "∩" being only partial: Whereas the latter is both $\psi^d_>$-computable and $\vartheta^d_>$-computable, union "∪" lacks $\mathring{\theta}^d_<$-continuity and $\vartheta^d_<$-continuity although closed complement is $(\psi^d_> \to \mathring{\theta}^d_<)$-computable, $(\mathring{\theta}^d_< \to \psi^d_>)$-computable, $(\vartheta^d_< \to \vartheta^d_>)$-computable, and $(\vartheta^d_> \to \vartheta^d_<)$-computable.

P r o o f.

$\prod^n \mathring{\theta}^d_< \longrightarrow \mathring{\theta}^d_<$. Given $\theta^d_<$-names of R_i° for $i = 1, \ldots, n$, $\theta^d_<$-compute $U := \bigcap R_i^\circ$ by virtue of [12, Corollary 5.1.18.1]. Then, by virtue of Lemma 4.2c), $U = R^\circ$, where $R := \bigcap R_i$.

$\prod^n \mathring{\theta}^d_> \to \mathring{\theta}^d_>$. Similarly.

$\prod^n \overline{\theta}^d_< \to \overline{\theta}^d_<$. Given $\theta^d_<$-names of U_i such that $\overline{U_i} = R_i$ for $i = 1, \ldots, n$, compute $U := \bigcap U_i$ as above. Then $\overline{U} = R$, where $R := \bigcap R_i$ by virtue of Lemma 4.3e) since by presumption the result is regular, i. e., $R = \overline{W}$ for some open W.

$\prod^n \vartheta^d_< \to \vartheta^d_<$. Proceeds similarly to the above proof of the $\vartheta^d_>$-computability of finite unions.

$\overline{\psi}^{\circ d}_{<} \times \overline{\psi}^{\circ d}_{<} \rightarrow \overline{\psi}^{\circ d}_{<}$. Recall [13, Theorem 5.2], where $\overline{\psi}^{\circ d}_{<}$ is proven to be uniformly equivalent to $\overline{\vartheta}^{\circ d}_{<}$. So w. l. o. g. let input R_1 and R_2 be given by lists $(\boldsymbol{a}_i)$ and $(\boldsymbol{b}_j)$ of rational vectors such that $R_1 = \overline{\left(\overline{\{\boldsymbol{a}_i : i\}}\right)^{\circ}}$ and similarly for R_2. Our algorithm searches, for each i in parallel (dove-tailing), for some j such that $\|\boldsymbol{a}_i - \boldsymbol{b}_j\| \leq 2^{-i}$ and, if found, reports $\boldsymbol{a}_i$. (Observe the seeming asymmetry in both inputs!)

Concerning its correctness, consider an arbitrary open ball $B(\boldsymbol{x}, \delta) \subseteq R := R_1 \cap R_2$; we shall argue that it contains some output $\boldsymbol{a}_i$ of the above algorithm as, by definition of denseness, this implies $\overline{\left(\overline{Q}\right)^{\circ}} \supseteq R$, where Q denotes the set of $\boldsymbol{a}_i$ reported. Indeed, as the input vectors $\boldsymbol{a}_i$ are dense in $R_1 \ni \boldsymbol{x}$, it holds $\|\boldsymbol{a}_i - \boldsymbol{x}\| \leq \delta$ for some i. Furthermore, $\boldsymbol{b}_j$ being dense in $R_2 \supseteq R^{\circ} \ni \boldsymbol{a}$, $\|\boldsymbol{b}_j - \boldsymbol{a}_i\| \leq 2^{-i}$ for some j. Hence the algorithm eventually outputs $\boldsymbol{a}_i$ to Q.

It remains to show $\overline{\left(\overline{Q}\right)^{\circ}} \subseteq R_2$ since "$\subseteq R_1$" holds trivially (only $\boldsymbol{a}_i$ are reported anyway). So let $B(\boldsymbol{x}, \delta) \subseteq \overline{Q}$; we shall argue that $B(\boldsymbol{x}, \delta)$ contains some $\boldsymbol{b}_j$ as this proves the $\boldsymbol{b}_j$ to be dense in $\overline{Q}$ from which the claim follows as $\overline{R_2^{\circ}} = R_2$ by regularity. Indeed Q being dense in $B(\boldsymbol{x}, \delta)$, $\|\boldsymbol{a}_i - \boldsymbol{x}\| \leq \delta/2$ for some $\boldsymbol{a}_i \in Q$. In fact we may presume $i \geq \log(2/\delta)$ because removing the finite initial part does not affect Q's denseness. From $\boldsymbol{a}_i$ having been reported by the above algorithm, it follows $\|\boldsymbol{a}_i - \boldsymbol{b}_j\| \leq 2^{-i} \leq \delta/2$ for some j. Hence $\boldsymbol{b}_j \in B(\boldsymbol{x}, \delta)$ which was to be shown.

$\prod^n \psi^d_{>} \rightarrow \psi^d_{>}$. [12, Theorem 5.1.13.2].

$\prod^n \overline{\psi}^{\circ d}_{>} \rightarrow \overline{\psi}^{\circ d}_{>}$. Let closed sets A_i such that $\overline{A_i^{\circ}} = R_i$ be given by respective $\psi^d_{>}$-names. $\psi^d_{>}$-compute $A := \bigcap A_i$ by means of [12, Theorem 5.1.13.2]. Then, for $R := \bigcap R_i = \overline{W}$ with open W,

$$\overline{A^{\circ}} = \overline{\bigcap A_i^{\circ}} = \bigcap \overline{A_i^{\circ}} = R \quad \text{(see Lemmas 4.2.c) and 4.3.e).}$$

$\prod^n \overline{\theta}^d_{>} \rightarrow \overline{\theta}^d_{>}$. Let open sets U_i such that $\overline{U_i} = R_i$ be given by respective $\theta^d_{>}$-names ($i = 1, \dots, n$). $\theta^d_{>}$-compute $U := \bigcap U_i$ by means of [12, Corollary 5.1.18.1]. Then, for $R := \bigcap R_i = \overline{W}$ with open W, $\overline{U} = R$ according to Lemma 4.3e).

$\prod^n \vartheta^d_{>} \rightarrow \vartheta^d_{>}$. Given respective enumerations of $Q_i := \mathbb{Q}^d \setminus R_i$ ($i = 1, \dots, n$) their union enumerates $\bigcup Q_i = \mathbb{Q}^d \setminus R$, where $R := \bigcap R_i$.

$\prod^n \overline{\vartheta}^d_{>} \rightarrow \overline{\vartheta}^d_{>}$. If $\overline{Q_i} = \overline{\mathbb{R}^d \setminus R_i}$, then $Q := \bigcup Q_i$ satisfies $\overline{Q} = \overline{\mathbb{R}^d \setminus R}$, where $R := \bigcap R_i$: Lemma 4.2g).

$\overline{\vartheta}^d_{<} \times \vartheta^d_{<} \rightarrow \overline{\vartheta}^d_{<}$. Given enumerations of $Q_1 \subseteq \mathbb{Q}^d$ such that $\overline{Q_1} = R_1$ and of $Q_2 := \mathbb{Q}^d \cap R_2^{\circ}$; compute straightforwardly $Q := Q_1 \cap Q_2$. Then obviously $\overline{Q} = R := R_1 \cup R_2$.

$\overline{\vartheta}^d_{<} \times \overline{\theta}^d_{<} \rightarrow \overline{\vartheta}^d_{<}$. Given $Q_1 \subseteq \mathbb{Q}^d$ such that $\overline{Q_1} = R_1$ and a $\theta^d_{<}$-name of some open U_2 such that $\overline{U_2} = R_2$; for each $\boldsymbol{q} \in Q_1$, test whether $\boldsymbol{q} \in U_2$ (a semi-decidable property according to Lemma 4.1c)) and, if yes, report $\boldsymbol{q}$. The resultingly enumerated set Q obviously has $\overline{Q} = R := R_1 \cap R_2$.

$\psi^d_{<} \times \overline{\theta}^d_{<} \rightarrow \psi^d_{<}$. Given a sequence $\boldsymbol{x}_i \in R_1$ dense in R_1 and a $\theta^d_{<}$-name of U_2 such that $\overline{U_2} = R_2$; for each i test (Lemma 4.1c)) whether $\boldsymbol{x}_i \in U_2$ and, if yes, report $\boldsymbol{x}_i$. This output is a $\psi^d_{<}$-name of $R := R_1 \cap R_2$.

$\overline{\vartheta}^d_{<} \times \overline{\vartheta}^d_{<} \not\rightarrow \psi^d_{<}$. For $d = 1$ let $R_1 = [0, 1]$ be given by an enumeration of the set Q_1 consisting of all rational vectors within $(0, 1)$ having (in reduced form) only *even* denominators; similarly Q_2 for $R_2 = [0, 1]$ with *odd* denominators. As $\overline{Q_i} = R_i$, these obviously *are* valid according $\overline{\vartheta}_{<}$-names. Were their intersection $\psi^d_{<}$-continuously dependent upon this input, then one could assert the open rational interval $(0, 1)$ to meet $R = R_1 \cap R_2$ knowing only finite initial parts of Q_1 and Q_2. But as Q_1 and Q_2 are disjoint, these finite sequences are easily extended to valid $\overline{\vartheta}_{<}$-names of disjoint regular sets R_1' and R_2'. This contradicts the output $(0, 1)$ to meet $R' = R_1' \cap R_2'$. □

3.1 Image and pre-image

When considering image and pre-image of regular sets under real functions, already Proposition 1.1 revealed that these functions better be open in the sense of mapping open sets to open sets. For obtaining *computable* image and pre-image operators, it is quite natural to require this mapping to be effective in the sense that, from a list of open rational balls exhausting U, one can computably determine a similar list for $f[U]$.

Definition 3.5 Call an open function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *effectively open* if the mapping $U \longmapsto f[U]$ from $\mathfrak{O}^n$ to $\mathfrak{O}^m$ is $(\theta^n_{<} \rightarrow \theta^m_{<})$-computable.

Let us point out that effectivized classical Open Function Theorems such as [6, Corollary 4.4] yield a rich variety of effectively open functions.

Whereas computability of the pre-image of closed sets in general need not hold under arbitrary computable functions [12, Exercise 5.1.6], the situation is different for computable effectively open functions:

Proposition 3.6 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be effectively open. Then, closed pre-image of closed sets*

$$\mathfrak{A}^m \ni A \longmapsto \overline{f^{-1}[A]} \in \mathfrak{A}^n$$

is $(\psi^m_< \to \psi^n_<)$-computable. $(\psi^m_> \to \psi^n_>)$-computability of $A \longmapsto f^{-1}[A]$ holds for computable f.

Proof. The second claim is [12, Theorem 6.2.4.2]. For the first one, recall [12, Lemma 5.1.10] that a $\psi^m_<$-name for closed $A \subseteq \mathbb{R}^m$ can equivalently be considered as an exhaustive list of all open rational balls $U_i \subseteq \mathbb{R}^m$ intersecting A. For each i, employ *dove-tailing* to check through all open rational balls $V_j \subseteq \mathbb{R}^n$ whether $f[V_j] \supseteq \overline{U_i}$ holds and, if yes, output V_j. Correctness of this algorithm follows from the lemma below. □

Lemma 3.7

a) *The above test is semi-decidable.*

b) *If V_j is reported, then it intersects $f^{-1}[A]$.*

c) *All open rational balls intersecting $f^{-1}[A]$ are reported.*

d) *An open ball V intersects $f^{-1}[A]$ iff it intersects $\overline{f^{-1}[A]}$.*

Proof.

a) By presumption, the open set $f[V_j]$ is $\theta^m_<$-computable. $\overline{U_i}$ being effectively compact, the claim now follows from Lemma 4.1b).

b) As $\emptyset \neq A \cap U_i \ni \boldsymbol{y}$ and $f[V_j] \supseteq \overline{U_i} \ni \boldsymbol{y}$, $\boldsymbol{y} = f(\boldsymbol{x})$ for some $\boldsymbol{x} \in V_j$. Hence $\boldsymbol{x} \in f^{-1}[\{\boldsymbol{y}\}] \subseteq f^{-1}[A]$, that is, V_j intersects $f^{-1}[A]$ in $\boldsymbol{x}$.

c) Let $V_j \subseteq \mathbb{R}^n$ denote an open rational ball intersecting $f^{-1}[A]$ in, say, $\boldsymbol{x} \in \mathbb{R}^n$, i. e., $\boldsymbol{y} := f(\boldsymbol{x}) \in A \subseteq \mathbb{R}^m$. Consider the open set $f[V_j] \ni \boldsymbol{y}$. Open rational balls forming a topological basis, $\boldsymbol{y} \in U_i \subseteq \overline{U_i} \subseteq f[V_j]$ for some open rational ball $U_i \subseteq \mathbb{R}^m$. This U_i thus intersects A (namely in $\boldsymbol{y}$) and hence must occur in the $\psi^m_<$-input for A. Since $f[V_j] \supseteq \overline{U_i}$, the above algorithm will indeed eventually report V_j.

d) Apply Lemma 4.3a) to $S := f^{-1}[A]$. □

Concerning computability of pre-image of regular sets, we now have:

Theorem 3.8 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be computable and open. Then the pre-image $\mathfrak{R}^m \ni R \longmapsto f^{-1}[R] \in \mathfrak{R}^n$ is*

$(\mathring{\theta}^m_< \to \mathring{\theta}^n_<)$*-computable;*	$(\psi^m_> \to \psi^n_>)$*-computable;*
$(\overline{\theta}^m_< \to \overline{\theta}^n_<)$*-computable;*	$(\overline{\mathring{\psi}}^m_> \to \overline{\mathring{\psi}}^n_>)$*-computable;*
$(\vartheta^m_< \to \overline{\vartheta}^n_<)$*-discontinuous in general;*	$(\vartheta^m_> \to \overline{\vartheta}^n_>)$*-discontinuous in general.*

In case f is furthermore effectively open, pre-image is also

$(\psi^m_< \to \psi^n_<)$*-computable;*	$(\mathring{\theta}^m_> \to \mathring{\theta}^n_>)$*-computable;*
$(\overline{\mathring{\psi}}^m_< \to \overline{\mathring{\psi}}^n_<)$*-computable;*	$(\overline{\theta}^m_> \to \overline{\theta}^n_>)$*-computable.*

Here the proofs mostly rely on known results concerning computability of pre-image of open and closed sets ([12, Theorem 6.2.4]) combined with commutation properties of set operations pre-image, topological closure, and topological interior, collected in Lemma 4.4.

Proof.

$\mathring{\theta}^m_< \to \mathring{\theta}^n_<$. Observe that $\big(f^{-1}[R]\big)^\circ = f^{-1}\big[R^\circ\big]$ according to Lemmas 4.4a) and 4.4b). Now use [12, Theorem 6.2.4.1] to $(\theta^m_< \to \theta^n_<)$-compute $R^\circ \longmapsto f^{-1}[R^\circ]$.

$\overline{\theta}^m_< \longrightarrow \overline{\theta}^n_<$. Let $R = \overline{U}$ with $U \subseteq \mathbb{R}^m$ open. Then $U \longmapsto V := f^{-1}[U]$ is $(\theta^m_< \to \theta^n_<)$-computable. Now observe $\overline{f^{-1}[U]} = f^{-1}\big[\,\overline{U}\,\big]$ according to Lemmas 4.4a) and 4.4b).

$\vartheta^m_< \not\to \overline{\vartheta}^n_<$. Consider computable open $f : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto \sqrt{2}t$.

$\psi^m_< \to \psi^n_<$. This follows from Proposition 3.6.

$\overline{\mathring{\psi}}^m_< \to \overline{\mathring{\psi}}^n_<$. Given closed A s. t. $R = \overline{A^\circ}$, compute according to Proposition 3.6 the closed set $B := f^{-1}[A]$ and observe that Lemmas 4.4a) and 4.4b) assert $\overline{B^\circ} = f^{-1}[R]$.

The dual claims, with respect to 'negative' representations that is, follow from the above by virtue of $f^{-1}[R] = \overline{\mathbb{R}^n \setminus f^{-1}\big[\overline{\mathbb{R}^m \setminus R}\big]}$ and Theorem 3.1. □

Finally addressing computability of set image, it holds:

Theorem 3.9 *Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be computable and open. Consider the image operator $R \longmapsto g[R]$ on* (*arbitrarily*) *bounded regular sets. This mapping is*

- $(\mathring{\theta}^n_< \to \mathring{\theta}^m_<)$*-discontinuous in general;*
- $(\overline{\theta}^n_< \to \overline{\theta}^m_<)$*-computable for effectively open g;*
- $(\psi^n_< \to \psi^m_<)$*-computable;*
- *in general* $(\vartheta^n_< \to \overline{\vartheta}^m_<)$*-discontinuous and* $(\vartheta^n_> \to \overline{\vartheta}^m_>)$*-discontinuous;*
- $(\psi^n_> \to \psi^m_>)$*-computable on $R \subseteq [-1, +1]^n$.*

Similarly to Proposition 1.1e) *rather than* f)*, the second and third claims also hold on unbounded $R \in \mathfrak{R}^n$, provided g is in addition closed.*

Proof.

$\mathring{\theta}^n_< \not\to \mathring{\theta}^m_<$. Consider $g : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^2$, identified with $g : \mathbb{R} \longrightarrow \mathbb{R}^2$, and for $\varepsilon \geq 0$ let

$$R_\varepsilon := \left\{r \exp(i\varphi) : \frac{1}{4}\pi \leq \varphi \leq \frac{3}{4}\pi \text{ and } \frac{1}{2} \leq r \leq 1\right\} \cup \left\{r \exp(i\varphi) : \frac{5}{4}\pi \leq \varphi \leq \frac{7}{4}\pi \text{ and } 1 + \varepsilon \leq r \leq 2\right\}.$$

$\overline{\theta}^n_< \to \overline{\theta}^m_<$. Let $R = \overline{U}$ be given by a $\theta^n_<$-name for open U. Exploiting effective openness, $\theta^m_<$-compute $V := g[U]$. Then $\overline{V} = g[R]$ holds: by virtue of Lemma 4.4d) as g is presumed computable and R is a bounded regular set; but, by virtue of Lemmas 4.4a and 4.4c), also under the alternative provision that $R \in \mathfrak{R}^n$ is arbitrary and g is in addition closed.

$\psi^n_< \to \psi^m_<$. Use [12, Theorem 6.2.4.3] to $\psi^m_<$-compute $\overline{g[R]}$ and observe that this coincides with $g[R]$ according to Lemma 4.4 – either applying Claim d) or Claims a) and c).

$\vartheta^m_< \not\to \overline{\vartheta}^n_<$ and $\vartheta^m_> \not\to \overline{\vartheta}^n_>$. Consider computable open closed $g : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto \sqrt{2}t$.

$\psi^n_> \to \psi^m_>$. [12, Theorem 6.2.4.4]. □

3.2 Convex hull

The *convex hull* $\text{chull}(S)$ of a set $S \subseteq \mathbb{R}^d$ is the smallest convex set containing S, i. e.,

$$\text{chull}(S) := \bigcap\{C \subseteq \mathbb{R}^d : S \subseteq C \text{ and } C \text{ is convex}\},$$

where $C \subseteq \mathbb{R}^d$ is called *convex* iff $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in C$ for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and $0 < \lambda < 1$.

Proposition 3.10

a) *The convex hull of a closed set does not need to be closed.*

b) *The convex hull of a bounded closed* (*i. e., a compact*) *set is compact.*

c) *The convex hull of a bounded regular set is* (*bounded and*) *regular.*

d) *A non-empty closed convex set is regular iff full-dimensional iff it contains an interior point.*

P r o o f . By virtue of a Theorem of Carathéodory [5, Paragraph 2.3.4],

(1) $$\mathrm{chull}(A) = \big\{ \sum_{i=0}^{d} \lambda_i \boldsymbol{x}_i : \boldsymbol{x}_0, \dots, \boldsymbol{x}_d \in A,\ \lambda_0, \dots, \lambda_d \geq 0, \text{ and } \sum_{i=0}^{d} \lambda_i = 1 \big\},$$

that is, the image under $H : \mathbb{R}^{(d+1)d} \times \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$, $(\boldsymbol{x}_0, \dots, \boldsymbol{x}_d, \lambda_0, \dots, \lambda_d) \longmapsto \sum_{i=0}^{d} \lambda_i \boldsymbol{x}_i$ of the set $(\prod^{d+1} A) \times \Lambda_{d+1}$, where $\Lambda_{d+1} := \big\{(\lambda_0, \dots, \lambda_d) : \lambda_i \geq 0, \text{ and } \sum \lambda_i = 1\big\}$.

a) $A := \big\{(t, \mathrm{e}^t) : t \leq 0\big\}$ is a closed set in $\mathbb{R}^2$, but its convex hull contains points $(x, y) = (-1, 1-\varepsilon)$ without actually reaching $(-1, 1)$.

b) This follows from the above observations that, if $A \subseteq \mathbb{R}^d$ is compact, $\mathrm{chull}(A)$ is the image under continuous function H of the compact set $(\prod^{d+1} A) \times \Lambda_{d+1}$.

c) This follows from Proposition 3.10d).

d) For non-empty closed convex $C \subseteq \mathbb{R}^d$ consider the following three claims: (i) $\dim(C) = n$, (ii) $C^\circ \neq \emptyset$ (iii) $\overline{C^\circ} = C$.

(iii)⇒(ii). $\emptyset \neq C = \overline{C^\circ}$ by (iii), thus $\emptyset \neq C^\circ$.

(ii)⇒(iii). Fix $\boldsymbol{p} \in C^\circ$ and $r > 0$ such that $B(\boldsymbol{p}, r) \subseteq C$. Let $\boldsymbol{x} \in C$, we show $\boldsymbol{x} \in \overline{C^\circ}$. This proves $C \subseteq \overline{C^\circ}$, the other inclusion follows from Lemma 4.2j). W. l. o. g. $\boldsymbol{x} = \boldsymbol{0}$, otherwise consider $C' := -\boldsymbol{x} + C$. By convexity, $\boldsymbol{x}_n := \frac{1}{n}\boldsymbol{p} \in C$. Even more, $B(\boldsymbol{x}_n, r/n) \subseteq C$. Thus $\boldsymbol{x}_n \in C^\circ$ for all $n \in \mathbb{N}$, hence $\boldsymbol{x} = \lim \boldsymbol{x}_n \in \overline{C^\circ}$.

(ii)⇒(i). Let $B(\boldsymbol{p}, r) \subseteq C \subseteq \mathbb{R}^d$. Then by monotonicity, $d = \dim B(\boldsymbol{p}, r) \leq \dim C \leq \dim \mathbb{R}^d = d$.

(i)⇒(ii). By presumption, the affine hull of C spans whole $\mathbb{R}^d$. W. l. o. g. $m\boldsymbol{0} \in C$, i. e., one may consider the linear rather than affine hull. Choose a basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_d \in C$ for $\mathbb{R}^d$ and let $\boldsymbol{b}_0 := \boldsymbol{0} \in C$. By convexity, the entire simplex S spanned by $\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ is contained in C. Based on the independence of the $\boldsymbol{b}_i$, it is easy to see that already S has positive volume and non-empty interior. □

Theorem 3.11 *The convex hull of bounded closed sets* $\mathfrak{A}^d \ni A \longmapsto \mathrm{chull}(A) \in \mathfrak{A}^d$ *is* $(\psi^d_< \to \psi^d_<)$*-computable. Restricted to closed subsets of* $[-1, +1]^d$*, it is furthermore* $(\psi^d_> \to \psi^d_>)$*-computable. Convex hull on (arbitrarily) bounded regular sets is in addition*

$$(\mathring{\theta}^d_< \to \mathring{\theta}^d_<)\text{-computable;} \qquad (\overline{\theta}^d_< \to \overline{\theta}^d_<)\text{-computable;}$$
$$(\vartheta^d_< \to \vartheta^d_<)\text{-computable;} \qquad (\overline{\vartheta}^d_< \to \overline{\vartheta}^d_<)\text{-computable;}$$

but $(\overline{\mathring{\psi}}{}^d_< \to \overline{\mathring{\psi}}{}^d_<)$*-discontinuous and, even on bounded regular subsets of* $\mathbb{R}^d$*, in general* $(\psi^d_> \to \mathring{\theta}^d_>)$*-discontinuous.*

P r o o f. Recall equation (1) from the proof of Proposition 3.10 with the function H and set Λ_{d+1} defined there.

(i) First, observe that H is a computable real function.

(ii) Second, the (constant) set $\Lambda_{d+1} \subseteq [-1, +1]^{d+1}$ is easily seen to be compact, $\psi^{d+1}_<$-computable, and $\psi^{d+1}_>$-computable.

(iii) Third, a variation of [12, Exercise 5.1.28] reveals $A \longmapsto \prod^{d+1} A$ to be both $(\psi^d_< \to \psi^{(d+1)d}_<)$-computable and $(\psi^d_> \to \psi^{(d+1)d}_>)$-computable.

(iv) Moreover, $\prod^{d+1} A$ is bounded iff A is; and $\prod^{d+1} A \subseteq [-1, +1]^{(d+1)d}$ for $A \subseteq [-1, +1]^d$.

Hence by virtue of [12, Theorem 6.2.4.3] and Proposition 3.10b),

$$\mathrm{chull}(A) = \overline{H\big[\big(\prod^{d+1} A\big) \times \Lambda_{d+1}\big]} = H\big[\big(\prod^{d+1} A\big) \times \Lambda_{d+1}\big]$$

is $\psi^d_<$-computable from a $\psi^d_<$-name of A. In case $A \subseteq [-1, +1]^d$, it is furthermore $\psi^d_>$-computable according to [12, Theorem 6.2.4.4].

Since, on convex regular output, $\mathring{\theta}^d_< \equiv \psi^d_<$ by virtue of [13, Theorem 4.12a], $(\psi^d_< \to \psi^d_<)$-computability implies $(\psi^d_< \to \mathring{\theta}^d_<)$-computability of the convex hull on bounded regular input. Supplying stronger input or demanding weaker output in the sense of Figure 1 is obviously no problem anyway.

Already for $d = 1$, one can easily construct examples for $(\psi^d_> \to \mathring{\theta}^d_>)$-discontinuity: Let $R = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, +1] \subseteq \mathbb{R}$ and consider an according $\psi_>$-name, that is, a list of open rational intervals (α_i, β_i) covering exactly $\mathbb{R} \setminus R$. Were chull continuous in the claimed sense, one could deduce from a finite initial segment $i = 1, \dots, N$ of this information the open rational interval $(2,3)$ to not entirely belong to $\operatorname{chull}(R)$. Now modify R and consider $R' := R \cup [\gamma, \gamma+1]$, where $\gamma := 3 + \max_i \beta_i$. Then the above initial segment is, up to $i = N$, also valid for R'; but $\operatorname{chull}(R')$ does contain $(2,3)$, contradiction.

Concerning $(\overline{\psi^{\circ d}_<} \to \overline{\psi^{\circ d}_<})$-discontinuity even on regular subsets of $[-1,+1]^d$, recall that according input/output encoding for some $R \in \mathfrak{R}^d$ is uniformly equivalent to the enumeration of a set Q of rational vectors such that $R = \overline{(\overline{Q})^\circ}$; in formula: $\overline{\psi^{\circ d}_<} \equiv \overline{\vartheta^{\circ d}_<}$. So consider the effectively bounded one-dimensional regular set $R_1 := [-1, -\frac{1}{2}] \cup [+\frac{1}{2}, +1]$ and an according $\vartheta_<$-name Q_1. Upon input of the latter, a presumed $(\overline{\vartheta^\circ_<} \to \overline{\vartheta^\circ_<})$-computation of $\operatorname{chull}(R)$ will output a name for $R' = [-1,+1]$ and in particular some rational $\boldsymbol{q}'_1 \in (-\frac{1}{2}, +\frac{1}{2})$. Having up to that moment read only finitely many $\boldsymbol{q}_i \in R_1^\circ$, consider $R_2 := [-1, -\frac{2}{3}] \cup [+\frac{2}{3}, +1]$ and observe that the initial segment read so far permits continuation to a valid $\overline{\vartheta^\circ_<}$-name Q_2 for R_2 as in this representation, finitely many 'wrong' points do not matter by definition. Thus for this continued input Q_2 of R_2, the presumed computation will again eventually report some $\boldsymbol{q}'_2 \in (-\frac{1}{2}, 0)$ because, still, $\operatorname{chull}(R_2) = [-1,+1]$. Then repeating the argument above, consider $R_3 := [-1, -\frac{3}{4}] \cup [+\frac{3}{4}, +1]$ until output of some $\boldsymbol{q}'_3 \in (0, \frac{1}{2})$; and so on. In the end (after 'infinite' time), this will lead to a sequence $\boldsymbol{q}'_i$ of rational vectors dense (at least) in $[-\frac{1}{2}, +\frac{1}{2}]$ whereas the overall input Q has accumulation points only -1 and $+1$, i. e., Q encodes $R = \overline{(\overline{Q})^\circ} = \overline{\{-1,+1\}^\circ} = \emptyset$ with $\operatorname{chull}(R) = \emptyset$ contradicting the output. □

4 Tools

This section collects some technical tools which proofs of the above results rely on but which at that place would have interrupted the train of thoughts. Here is, for instance, a result concerning several properties of sets to be *semi-decidable* in the sense that some Type-2 Machine exists which, upon input of an encoding γ of the set as specified, terminates iff this set has the property under consideration. Such properties are formalized in TTE as *γ-r. e. open*; cf. [12, Definitions 2.4.1.1 and 3.1.3.2]. Claim a) in the lemma below for instance means that, given $N \in \mathbb{N}$ and a $\psi^d_>$-name of a closed subset A of $[-N,+N]^d$, it is semi-decidable whether "$A = \emptyset$" holds.

Lemma 4.1

a) $\{(\emptyset, N) : N \in \mathbb{N}\}$ *is* $(\psi^d_>, \nu_\mathbb{N})$*-r. e. open in* $\{(A,N) : A \subseteq [-N,+N]^d$ *closed*$\}$*, where* $\nu_\mathbb{N}$ *denotes the standard notation of natural numbers.*

b) $\{(A,U,N) : A \subseteq [-N,+N]^d$ *closed and* $A \subseteq U \subseteq \mathbb{R}^d$ *open*$\}$ *is* $(\psi^d_>, \theta^d_<, \nu_\mathbb{N})$*-r. e. open in* $\{(A,U,N) : A \subseteq [-N,+N]^d$ *closed and* $U \subseteq \mathbb{R}^d$ *open*$\}$.

c) $\{(\boldsymbol{x}, U) : \boldsymbol{x} \in U \subseteq \mathbb{R}^d$ *open*$\}$ *is* $(\varrho^d, \theta^d_<)$*-r. e. open in* $\{(\boldsymbol{x}, U) : \boldsymbol{x} \in \mathbb{R}^d,\ U \subseteq \mathbb{R}^d$ *open*$\}$*, where* ϱ^d *denotes the Cauchy representation of real vectors.*

Proof.

a) A $\psi^d_>$-name for A is well-known to be (equivalent to) a $[\varrho^d \to \varrho_<]$-name for A's Euclidean distance function [12, Lemmas 5.1.7 and 5.1.10]. In particular, it allows for effective approximation from below of this function at argument $\boldsymbol{x} = \boldsymbol{0}$, that is, a $\varrho_<$-name for the real number $r = \inf\{\|\boldsymbol{a}\| : \boldsymbol{a} \in A\}$ which, for $A = \emptyset$, is equal to infinity by convention; cf. also [12, Definition 4.1.21]. Conversely for $\emptyset \neq A \subseteq [-N,+N]^d$, $r \leq N$. It suffices to test whether "$N < r$" hold, an obviously r. e. property by [12, Definition 4.1.3].

b) Observe that $A \subseteq U$ iff $A \cap (\mathbb{R}^d \setminus U) = \emptyset$. So $U \longmapsto \mathbb{R}^d \setminus U =: B$ being $(\theta^d_< \to \psi^d_>)$-computable by [12, Definition 5.1.15.1] and $(A,B) \longmapsto A \cap B$ being $(\psi^d_> \times \psi^d_> \to \psi^d_>)$-computable [12, Theorem 5.1.13.2], the claim is reduced to a).

c) It is well-known that, from a ϱ^d-name of $\boldsymbol{x}$, one can both obtain a non-negative integer N such that $N \geq \|\boldsymbol{x}\|$ [12, Example 4.1.10] and a $\psi^d_>$-name of $A := \{\boldsymbol{x}\}$; see [12, Example 5.1.3.5]. Now apply b). □

4.1 Topology

For ease of reference, we here recall some facts from general topology, cf. e. g. [9]. For a topological space X and $M \subseteq X$, $\overline{M}$ means the smallest closed set containing M and M° the largest open set contained within M.

Lemma 4.2 *Let I denote an arbitrary index set and $G, H, M_i \subseteq X$, $i \in I$.*

a) $\bigcup_{i\in I} \overset{\circ}{M_i} \subseteq \big(\bigcup_{i\in I} M_i\big)^\circ$; e) $\bigcap_{i\in I} \overset{\circ}{M_i} \supseteq \big(\bigcap_{i\in I} M_i\big)^\circ$;

b) $\bigcap \overline{M_i} \supseteq \overline{\bigcap M_i}$; f) $\bigcup \overline{M_i} \subseteq \overline{\bigcup M_i}$;

c) $G^\circ \cap H^\circ = (G \cap H)^\circ$; g) $\overline{G} \cup \overline{H} = \overline{G \cup H}$;

d) $(X \setminus G)^\circ = X \setminus \overline{G}$; h) $\overline{X \setminus H} = X \setminus \overset{\circ}{H}$.

Let U denote an open and A a closed subset of X.

i) $U \subseteq \overset{\circ}{\overline{U}}$; k) $U = A^\circ \Rightarrow \overset{\circ}{\overline{U}} = U$;

j) $A \supseteq \overline{A^\circ}$; k) $A = \overline{U} \Rightarrow \overline{A^\circ} = A$.

From this, we conclude

Lemma 4.3

a) *Let $V \subseteq X$ be open and $S \subseteq X$ arbitrary such that $V \cap \overline{S} \neq \emptyset$. Then $V \cap S \neq \emptyset$.*

b) *Let $A, B \subseteq X$ be closed. Then $\overline{A^\circ} \cup \overline{B^\circ} = \overline{(A \cup B)^\circ}$.*

c) *Let $U \subseteq X$ be open, $Q \subseteq X$ arbitrary but dense in X. Then $\overline{U \cap Q} = \overline{U}$.*

d) *Let $n \in \mathbb{N}$, $U_i \subseteq X$ open such that $\overline{U_i} = X$ for $i = 1, \dots, n$. Then $\overline{\bigcap_{i=1}^n U_i} = X$.*

e) *Let $U_i, W \subseteq X$ be open for $i = 1, \dots, n$ and $\bigcap_{i=1}^n \overline{U_i} = \overline{W}$. Then $\overline{W} = \overline{\bigcap_{i=1}^n U_i}$.*

f) *Let X be a Baire space[2], $U_n, W \subseteq X$ open for $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \overline{W}$. Then $\overline{W} = \overline{\bigcap_n U_n}$.*

Proof.

a) Suppose $V \cap S = \emptyset$; i.e., $S \subseteq X \setminus V$. Then, by monotonicity, $\overline{S} \subseteq \overline{X \setminus V} = X \setminus V$ as V is open. Therefore $V \cap \overline{S} = \emptyset$, a contradiction.

b) "$\subseteq$" follows from Lemmas 4.2a) and 4.2g). For the reverse inclusion first suppose that $A \cup B = X$; equivalently: A contains the open set $X \setminus B$. Hence $A^\circ \supseteq X \setminus B$ as well as $\overline{A^\circ} \supseteq X \setminus B^\circ$ by monotonicity. This in turn is equivalent to $X = \overline{A^\circ} \cup B^\circ \subseteq \overline{A^\circ} \cup \overline{B^\circ}$. Since by presumption $X = \overline{X^\circ} = (A \cup B)^{\overline{\circ}}$, this yields the claim.

Now suppose the closed set $C := X \setminus (A \cup B)^\circ$ is non-empty. Consider closed set $A' := A \cup C$ to verify that $A' \cup B = (A \cup B) \cup C \supseteq (A \cup B)^\circ \cup X \setminus (A \cup B)^\circ = X$. So the above case applies and yields $X = \overline{A'^\circ} \cup \overline{B'^\circ}$. This in turn means $(A \cup C)^{\overline{\circ}} = \overline{A'^\circ} \supseteq X \setminus \overline{B^\circ}$. Again monotonicity implies $(A \cup C)^\circ = \overset{\circ}{\overline{(A \cup C)^\circ}} \supseteq X \setminus \overline{B^\circ}$ by virtue of Lemma 4.2k) and since the right hand side is already open. Equivalently, $X = \overline{B^\circ} \cup (A \cup C)^\circ \subseteq \overline{B^\circ} \cup A \cup C$ and thus $A \supseteq X \setminus (\overline{B^\circ} \cup C)$. We once again exploit monotonicity to obtain $A^\circ \supseteq X \setminus (\overline{B^\circ} \cup C)$ and conclude $X = A^\circ \cup \overline{B^\circ} \cup C$. This finally yields $(A \cup B)^\circ = X \setminus C \subseteq A^\circ \cup \overline{B^\circ} \subseteq \overline{A^\circ} \cup \overline{B^\circ}$, from which follows the claim by taking closures as the right hand side already is closed.

c) $U \cap Q \subseteq U$ yields $\overline{U \cap Q} \subseteq \overline{U}$ by monotonicity. For the reversed inclusion let $A := \overline{U \cap Q}$. The closed set $A' := A \cup (X \setminus U)$ contains $(Q \cap U) \cup (Q \setminus U) = Q$ so, again by monotonicity, $A' = \overline{A'} \supseteq \overline{Q} = X$. But $A' = X$ implies $U \subseteq A = \overline{U \cap Q}$.

d) This follows inductively from c), letting $U := U_d$ and $Q := \bigcap_{i=1}^{n-1} U_i$ which is dense by induction hypothesis.

e) Let $U_i' := U_i \cup (X \setminus \overline{W})$: These sets are open and dense in X according to Lemma 4.2g): $\overline{U_i'} = \overline{U_i} \cup \overline{(X \setminus \overline{W})} \supseteq \bigcap_i \overline{U_i} \cup (X \setminus \overline{W}) = X$. Hence, by virtue of d) and Lemmas 4.2g) and 4.2h), $X = \overline{\bigcap_i U_i'} = \overline{\bigcap_i U_i} \cup \big(X \setminus \overset{\circ}{\overline{W}}\big)$. Therefore $\overset{\circ}{\overline{W}} \subseteq \overline{\bigcap_i U_i}$ and, because of monotonicity plus Lemma 4.2l), $\overline{W} = \overline{\big(\overline{W}\big)^\circ} \subseteq \overline{\bigcap_i U_i}$. The other inclusion holds due to Lemma 4.2b).

f) Recall that in a Baire space by definition, the countable intersection of open dense subsets is dense again, that is, d) holds also for the case $n = \infty$. Now the proof of e) re-applies literally. □

[2] Like, for example, $X = \mathbb{R}^d$.

Lemma 4.4 *Let X, Y be topological spaces, $f : X \longrightarrow Y$, $G \subseteq X$, and $H \subseteq Y$.*

a) *If f is continuous, then $f[\overline{G}] \subseteq \overline{f[G]}$, $f^{-1}[\overline{H}] \supseteq \overline{f^{-1}[H]}$ and $f^{-1}[\mathring{H}] \subseteq (f^{-1}[H])^\circ$.*

b) *If f is open, then $f[\mathring{G}] \subseteq (f[G])^\circ$, $f^{-1}[\overline{H}] \subseteq \overline{f^{-1}[H]}$, and $f^{-1}[\mathring{H}] \supseteq (f^{-1}[H])^\circ$.*

c) *If f is closed, then $f[\overline{G}] \supseteq \overline{f[G]}$.*

d) *If f is continuous, Y is Hausdorff, and $\overline{G}$ is compact, then $f[\overline{G}] = \overline{f[G]}$.*

Proof.

a) $f^{-1}[\overline{H}]$ is a closed set (as f continuous) containing $f^{-1}[H]$; $\overline{f^{-1}[H]}$ on the other hand is by definition the smallest set with this property. That implies $f^{-1}[\overline{H}] \supseteq \overline{f^{-1}[H]}$. Analogously is $f^{-1}[H^\circ]$ an open subset of $f^{-1}[H]$; whereas $(f^{-1}[H])^\circ$ is the largest one.

For the first claim observe that, always,

$$f[f^{-1}[H]] \subseteq H \quad \text{and} \quad G \subseteq f^{-1}\big[f[G]\big]\,. \tag{2}$$

Consequently, $f^{-1}[\overline{f[G]}]$ contains G and is closed; hence it also contains $\overline{G}$. From $\overline{G} \subseteq f^{-1}[\overline{f[G]}]$, it now follows, by (2), $f[\overline{G}] \subseteq f\big[f^{-1}[\overline{f[G]}]\big] \subseteq \overline{f[G]}$.

b) Since $f[G^\circ]$ is open, it coincides with $(f[G^\circ])^\circ \subseteq (f[G])^\circ$. Applying this to $G := f^{-1}[H]$ yields, by (2) $f[f^{-1}[H]^\circ] \subseteq f\big[f^{-1}[H]\big]^\circ \subseteq H^\circ$ and therefore, again by (2) $(f^{-1}[H])^\circ \subseteq f^{-1}[f[f^{-1}[H]^\circ]] \subseteq f^{-1}[H^\circ]$. This also asserts the middle claim by virtue of $f^{-1}[Y \setminus H] = X \setminus f^{-1}[H]$ and Lemmas 4.2d) and 4.2h).

c) $f[\overline{G}]$ is a closed set containing $f[G]$ whereas $\overline{f[G]}$ is the smallest such set.

d) Any continuous function f maps a compact set like $\overline{G}$ to a compact set $f[\overline{G}]$.

In Hausdorff spaces, any compact set is closed. Now continue as in c) and combine with a). □

5 Conclusion

The present work investigated the effectivity of natural operators on regular sets: union, intersection, closed complement, image, and pre-image under suitable classes of functions. For each operation, its uniform computability was considered with respect to twelve basic representations identified in [13] which cover a vast number of others treated in literature.

Many of them turned out to exhibit deficiencies which limit their general applicability in Computability Theory: $\vartheta^d_<$, $\vartheta^d_>$, $\overline{\vartheta}^d_<$, and $\overline{\vartheta}^d_>$ are not robust [12]; $\mathring{\theta}^d_<$ and $\mathring{\theta}^d_>$ render binary union uncomputable, and so does $\psi^d_<$ for binary intersection. In fact, among those representation with respect to which most operations were proven computable, the join of $\overline{\theta}^d_<$ and $\psi^d_>$ is strongest. Calling this derived representation ξ^d, it follows from Theorems 3.2, 3.4, 3.8, 3.9, and 3.11:

Corollary 5.1 *The following operations on regular sets are ξ-computable:*

- *finite union;*
- *finite intersection (provided the result is regular again);*
- *pre-image under computable open functions;*
- *image under computable closed effectively open functions (on bounded arguments, the closedness condition may be omitted);*
- *convex hull of effectively bounded sets.*

Further on, ξ-computability of a regular set implies (even uniformly) computability of its distance function, that is, in the sense of [4, 1].

Indeed, ξ has been employed in [15] to prove computability of rather general *Nonlinear* Optimization Problems. However, also $\overline{\psi^{\circ d}_<} \equiv \overline{\vartheta}^{\circ d}_<$ and $\overline{\theta}^d_> \equiv \overline{\vartheta}^{\circ d}_>$ are remarkably interesting representations despite their weakness. Recall that a valid, say, $\overline{\vartheta}^{\circ d}_<$-name for R need not contain *all* rational vectors in R° and furthermore may even contain 'erroneous' points lying *outside* of R. This resembles Büchi's famous generalization of finite automata to infinite words [10], in that for both, computation is a truly *infinite* process: no finite initial segment can be exploited to prematurely pin down the eventual result. As a consequence, simple discontinuity does not in general induce uncomputability any more – an implication which otherwise Recursive Analysis has often been criticized for.

5.1 Future Research

For some basic representations, computability of function image is still unsettled: $\overline{\psi^{\circ}}^{d}_{<}$, $\overline{\psi^{\circ}}^{d}_{>}$, $\theta^{\circ d}_{>}$, and $\overline{\theta}^{d}_{>}$, to mention them. Also observe that for, say, the image $g[R]$ to be regular, it suffices that g be continuous open (and closed) only on R rather than whole $\mathbb{R}^n$. This raises the question of *partially* computability of function image for more general functions like, say, on $\{R \in \mathfrak{R}^n \ : \ g[R] \in \mathfrak{R}^m\} \ni R \longmapsto g[R]$ similarly to Theorem 3.4; analogous questions arise for pre-image.

Finally we wonder, in the spirit of [13], whether additional supply of information of opposite sign might affect the computability properties considered here.

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